Generating sets of Completely 0-Simple Semigroups

Robert Gray

University of St Andrews

Rank

Definition Let S be a semigroup and let T be a subset of S.

- The *rank* of S is the smallest number of elements needed in order to generate S:

$$\operatorname{rank}(S) = \min\{|A| : \langle A \rangle = S\}.$$

- The *relative rank* of S modulo T is the minimal number of elements of S that need to be added to T in order to generate the whole of S:

$$\operatorname{rank}(S : T) = \min\{|A| : A \subseteq S, \langle T \cup A \rangle = S\}.$$

Example: the structure of T_3

S_3

	$\{1,2\}$	$\{2,3\}$	$\{1,3\}$
$\{3\},\{1,2\}$	$[1,1,2][2,2,1]$	$[2,2,3],[3,3,2]$	$[1,1,3],[3,3,1]$
$\{1\},\{2,3\}$	$[1,2,2],[2,1,1]$	$[2,3,3],[3,2,2]$	$[1,3,3],[3,1,1]$
$\{2\},\{1,3\}$	$[1,2,1],[2,1,2]$	$[2,3,2],[3,2,3]$	$[1,3,1],[3,1,3]$

	$\{1\}$	$\{2\}$	$\{3\}$
$\{1,2,3\}$	$[1,1,1]$	$[2,2,2]$	$[3,3,3]$

Principal factors

Definition Let J be some $\mathcal{J}$ class of a semigroup S. Then the principal factor of S corresponding to J is the set $J^* = J \cup \{0\}$ with multiplication

$$s * t = \begin{cases} st & : \quad \text{if } \ s, t, st \in J \\ 0 & : \quad \text{otherwise.} \end{cases}$$

Definition A semigroup with zero is called 0-*simple* if $\{0\}$ and S are its only ideals.

Theorem If J is a $\mathcal{J}$ class of a semigroup S then J^* is either a 0-simple semigroup or else it is a zero semigroup.

Rees matrix semigroups

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- G - a finite group.

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- $P = (p_{\lambda i})$ a *regular* $\Lambda \times I$ matrix over $G \cup \{0\}$.
- $S = (I \times G \times \Lambda) \cup \{0\}$ with multiplication

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, gp_{\lambda j}h, \mu) & : \quad p_{\lambda j} \neq 0 \\ 0 & : \quad \text{otherwise} \end{cases}$$

$$(i, g, \lambda)0 = 0(i, g, \lambda) = 00 = 0.$$

Rees matrix semigroups

Theorem(The Rees Theorem) A semigroup S is completely 0-simple if and only if it is isomorphic to $\mathcal{M}^0[G; I, \Lambda; P]$ where G is a group and P is regular.

The BIG Problem

Problem Find a formula for the rank of an arbitrary completely 0-simple semigroup.

- What might we expect the value to depend on?

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- What might we expect the value to depend on?
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- $\mathrm{rank}(G)$.

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Problem Find a formula for the rank of an arbitrary completely 0-simple semigroup.

- $E(S)$ (‘contribution’ from the entries in the matrix).
- Remember that $(i, {p_{\lambda i}}^{-1}, \lambda)$ are idempotent

$$
\begin{aligned}
(i, {p_{\lambda i}}^{-1}, \lambda)(i, {p_{\lambda i}}^{-1}, \lambda) &= (i, {p_{\lambda i}}^{-1} p_{\lambda i} {p_{\lambda i}}^{-1}, \lambda) \\
&= (i, {p_{\lambda i}}^{-1}, \lambda).
\end{aligned}
$$

Special Cases

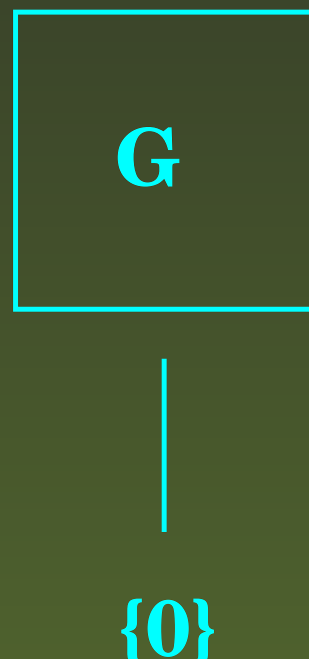
We will break the problem up and consider the following special cases:

- Groups.
- Rectangular bands.
- Rectangular 0-bands - $\mathcal{M}^0[\{e\}; I, \Lambda; P]$.
- Simple semigroups.
- Connected 0-simple semigroups.
- Brandt semigroups ($P \sim I$).

Groups

Lemma Let G be a finite group, then

$$\operatorname{rank}(\mathcal{M}^0[G; \{1\}, \{1\}; (1)]) = \operatorname{rank}(G).$$



Rectangular bands

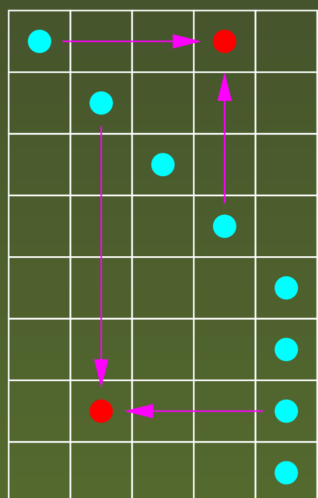
Definition $R_{mn} = \{1, \ldots, m\} \times \{1, \ldots, n\}$ with

$$(i, j)(k, l) = (i, l).$$

Proposition

$$\operatorname{rank}(R_{mn}) = \max\{m, n\}.$$

Proof



Rectangular 0-bands

Definition Let $I = \{1, 2, \ldots, m\}$ and $\Lambda = \{1, 2, \ldots, n\}$ be finite sets and let P be a regular $n \times m$ matrix of 0s and 1s. A *rectangular* 0-*band* is a semigroup $S = ZB_{mn} = (I \times \lambda) \cup \{0\}$ whose multiplication is given by

$$(i, \lambda)(j, \mu) = \begin{cases} (i, \mu) & : \quad \text{if } \; p_{\lambda j} = 1 \\ 0 & : \quad \text{if } \; p_{\lambda j} = 0 \end{cases}$$

$$(i, \lambda)0 = 0(i, \lambda) = 00 = 0.$$

Rectangular 0-bands

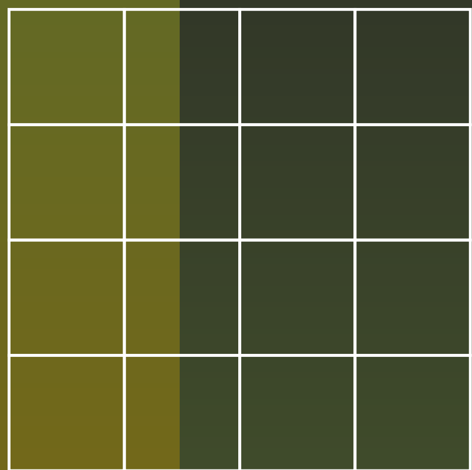


Figure 1:

$$P = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$$A = \{(1,1),(2,3),(3,4),(4,2)\}$$

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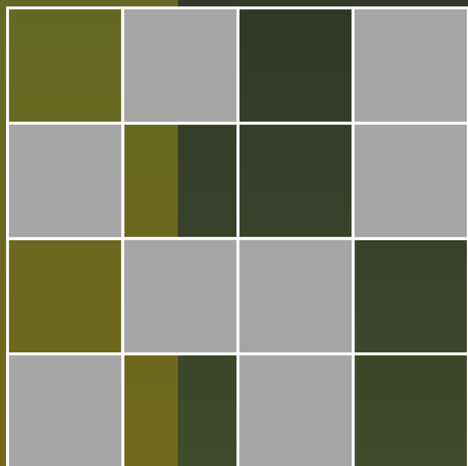

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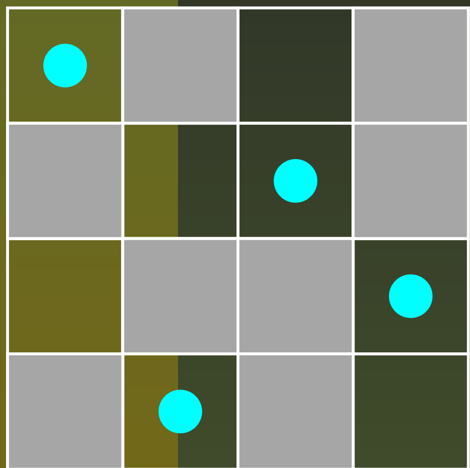

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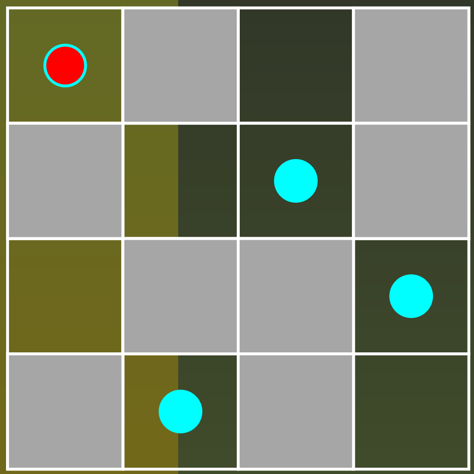

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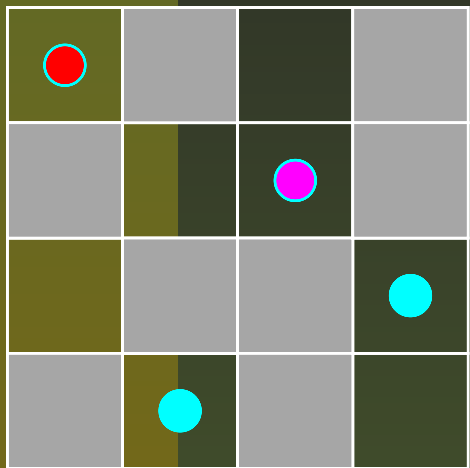

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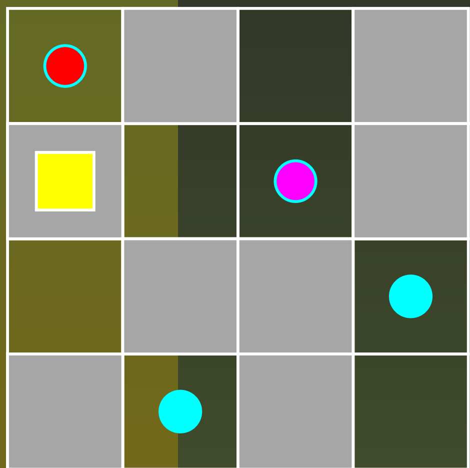

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$$p_{12} = 1$$

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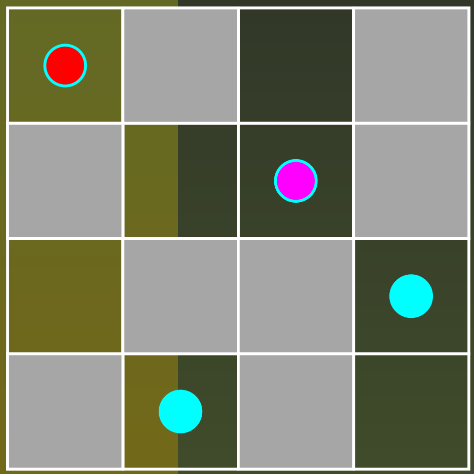

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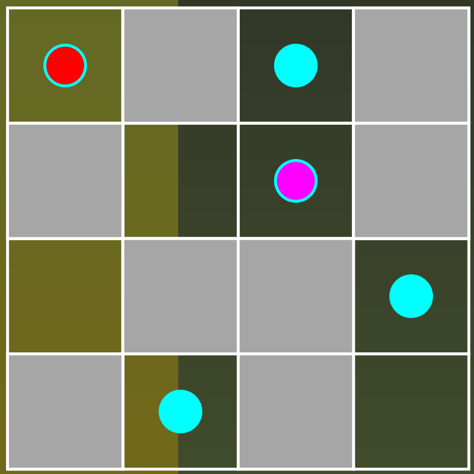

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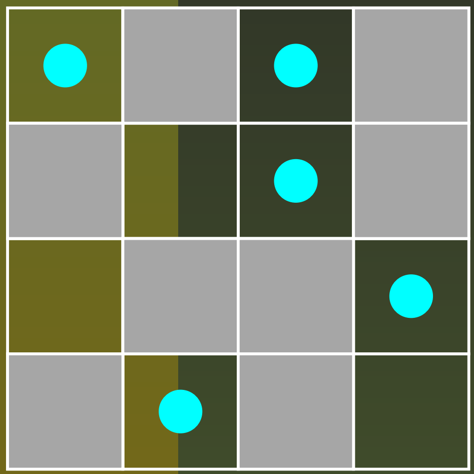

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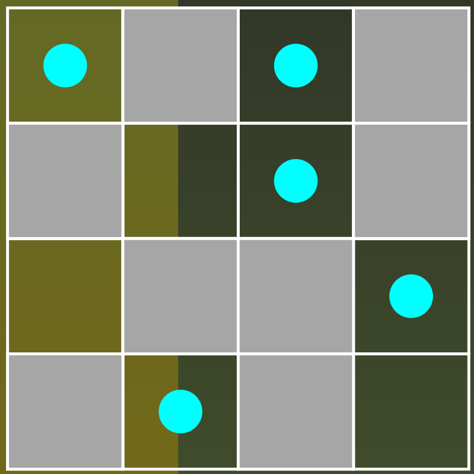

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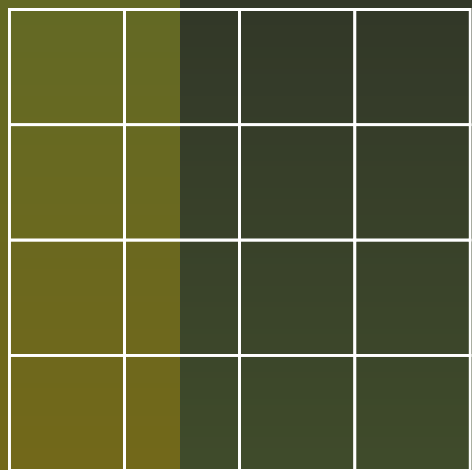

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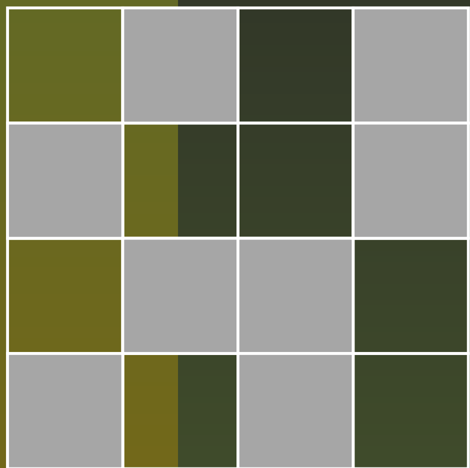

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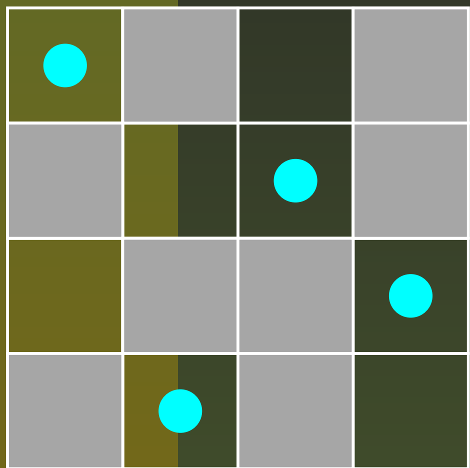

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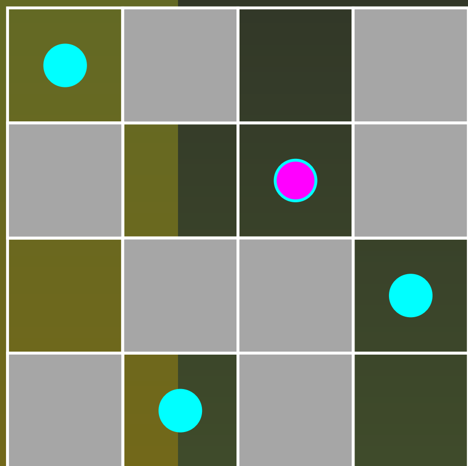

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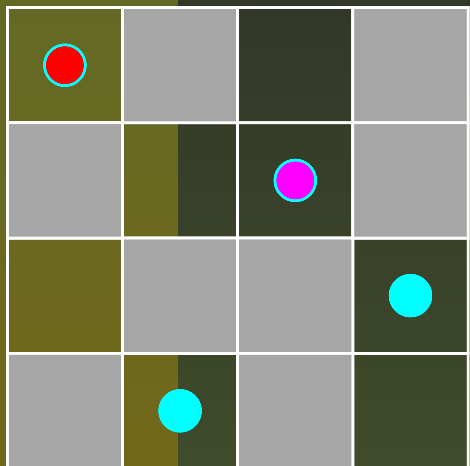

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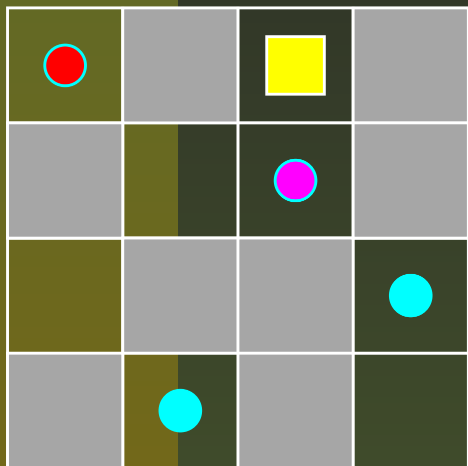

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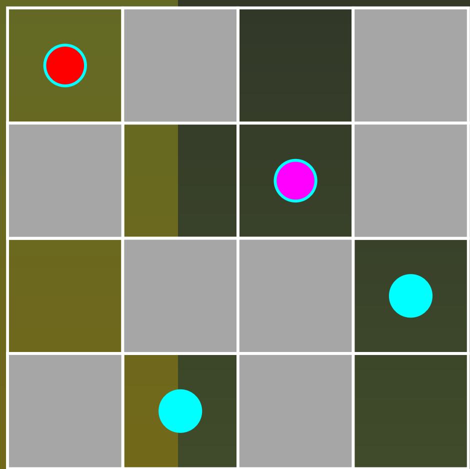

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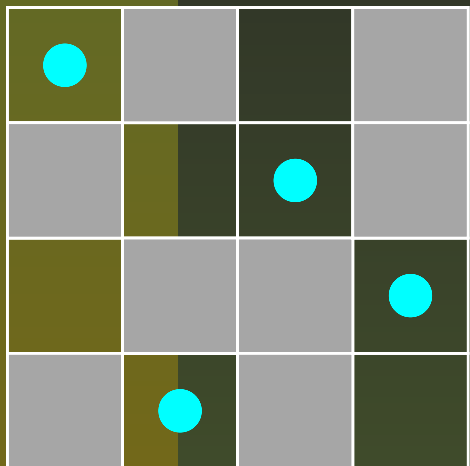

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Rectangular 0-bands

Theorem Let $S = ZB_{mn}$, be an $m \times n$ rectangular 0-band, then

$$\operatorname{rank}(S) = \max\{m, n\}.$$

Corollaries

Corollary If $S = \mathcal{M}^0[G; I, \Lambda; P]$ is idempotent generated then

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Corollary With

$$K(n,r) = \{\alpha \in T_n : |\mathrm{im}(\alpha)| \leq r\}, (2 \leq r \leq n-1)$$

we have

$$\begin{aligned} \mathrm{rank}(K(n,r)) &= \max(\binom{n}{r}, S(n,r)) \\ &= S(n,r). \end{aligned}$$

Simple

Theorem(NR,1994) Let $S = \mathcal{M}[G; I, \Lambda; P]$ be a finite Rees matrix semigroup with P in normal form. Then

$$\mathrm{rank}(S) = \max(|I|, |\Lambda|, \mathrm{rank}(G : H))$$

where $H = \langle P \rangle$.

Normal form

$$P = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & g_{22} & g_{23} & \dots & g_{2n} \\ 1 & g_{32} & g_{33} & \dots & g_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & g_{n2} & g_{n3} & \dots & g_{nn} \end{pmatrix}$$

Connected completely 0-simple semigroups

Definition Let $S = \mathcal{M}^0[G; I, \Lambda; P]$, then we let $\Gamma(S)$ be the graph with set of vertices $\{(i, \lambda) \in I \times \Lambda : H_{i\lambda}$ is a group$\}$ and (i, λ) adjacent to (j, μ) if and only if $i = j$ or $\lambda = \mu$.

Definition We say $S = \mathcal{M}^0[G; I, \Lambda; P]$ is connected if $\Gamma(S)$ is connected.

Example Connected.

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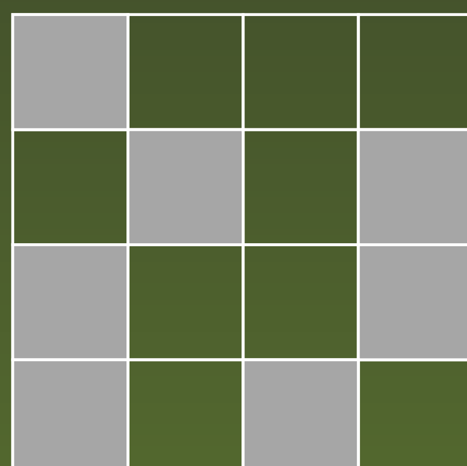

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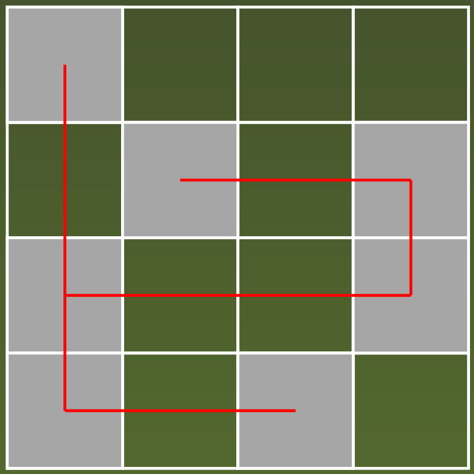

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Example Connected.
In particular $S = \mathcal{M}[G; I, \Lambda; P]$ (simple semigroups) are all connected.

Connected completely 0-simple semigroups

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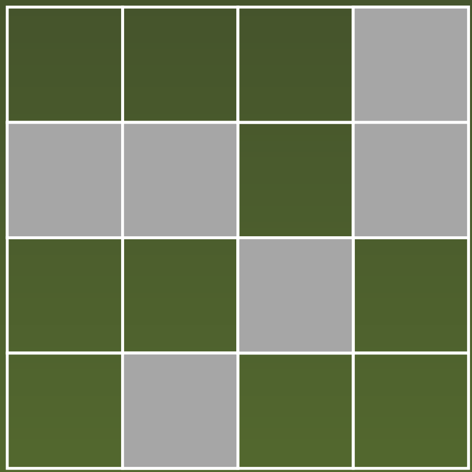

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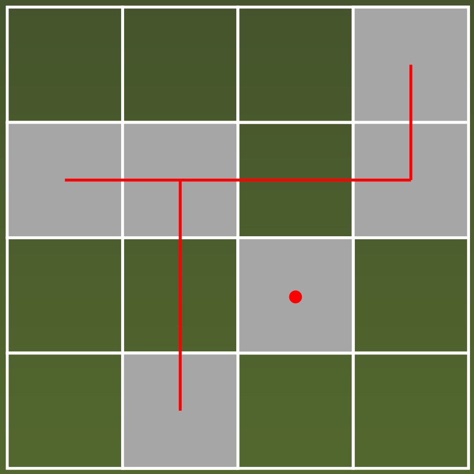

Connected

Theorem(NR,1994) Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a finite connected Rees matrix semigroup with regular matrix P (in normal form). Then

$$\mathrm{rank}(S) = \max(|I|, |\Lambda|, \mathrm{rank}(G : H))$$

where H is the subgroup of G generated by the non-zero entries in P.

Brandt Semigroup

Theorem(Howie,Gomes,1986) Let $B = B(G, \{1, \ldots, n\})$ be a Brandt semigroup, where G is a finite group of rank r. Then the rank of B (as an inverse semigroup) is $r + n - 1$.

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Proof ($\leq$)

$A = \{(1, g_1, 1), \ldots, (1, g_r, 1), (1, e, 2), (2, e, 3), \ldots, (n-1, e, n)\}$

($\geq$) Using graph theory.

What do we know?

	$G = \{e\}$	G arbitrary
Connected		
Disconnected		
Brandt		

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Brandt	$\max(n, n)$	$\max(n, n, r + n - 1)$

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Connected	$\max(\|I\|, \|\Lambda\|)$	$\max(\|I\|, \|\Lambda\|, \mathrm{rank}(G : H))$
Disconnected	$\max(\|I\|, \|\Lambda\|)$	?
Brandt	$\max(n, n)$	$\max(n, n, r + n - 1)$

General Formula

Theorem Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a finite Rees matrix semigroup with regular matrix P (in normal form) with connected components $C_1, \ldots, C_k$ and H_j the subgroup of G generated by all non-zero entries of C_j, for $j = 1, \ldots, k$. Then

$$\operatorname{rank}(S) = \max(|I|, |\Lambda|, r_{\min} + k - 1)$$

where

$$r_{\min} = \min_{(g_1, \cdots, g_k) \in \underbrace{G \times \ldots \times G}_{k}} (\operatorname{rank}(G : \cup_{j=1}^{k} {g_j}^{-1} H_j g_j)).$$