Semigroup presentations via boundaries in Cayley graphs [1]

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[1] (Research conducted while I was a research student at the University of St Andrews, under supervision of Nik Ruškuc.)

Semigroup presentations

Definition

- Presentation: $\langle A|R\rangle$ A - alphabet (abstract generators) $R \subseteq A^+ \times A^+$ set of pairs of words (defining relations)
- Defines the semigroup $S \cong A^+/\eta$ where η is the smallest congruence on A^+ containing R.
- S is finitely generated if A can be chosen to be finite.
- S is finitely presented if A and R can both be chosen to be finite.

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Facts

- Every finite semigroup is finitely presented (Cayley table).
- Not every finitely generated semigroup is finitely presented

$$\langle a, b \mid ab^i a = aba \; (i = 2, 3, \ldots)\rangle.$$

Presentations for subsemigroups

Let T be a subsemigroup of S.

In general...

1. S finitely generated $\not\Rightarrow$ T finitely generated.
2. S finitely presented and T finitely generated $\not\Rightarrow$ T finitely presented.

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Question

Can the condition $|S \setminus T| < \infty$ be replaced by something weaker?

Cayley Graphs

Definition

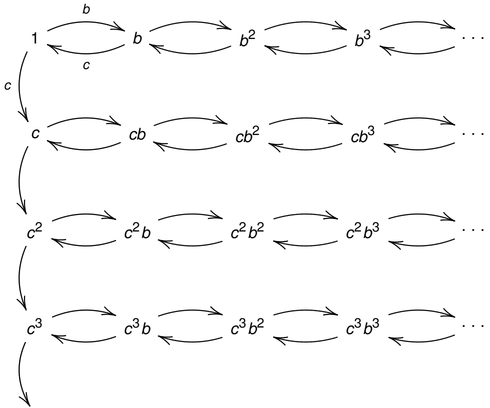
Let S be a semigroup generated by a finite set A

The right Cayley graph $\Gamma_r(A, S)$ has:

- Vertices: elements of S.
- Edges: directed and labelled with letters from A.

$$s \xrightarrow{a} t \Leftrightarrow sa = t$$

Bicyclic monoid $B = \langle b, c | bc = 1 \rangle$



$\Gamma_r(\{b, c\}, B)$

Semigroup boundaries

Definition

- Let T be a subsemigroup of S.
- The right boundary of T in S is the set of elements of T that *receive an edge from* $S \setminus T$ in the right Cayley graph of S:

$$\mathcal{B}_r(A, T) = (S \setminus T)A \cap T.$$

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- The left boundary of T in S is the set of elements of T that receive an edge from $S \setminus T$ in the left Cayley graph of S:

$$\mathcal{B}_l(A, T) = A(S \setminus T) \cap T.$$

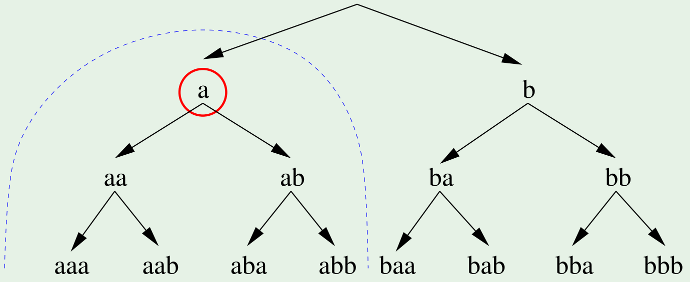
- The (two-sided) boundary is the union of the left and right boundaries:

$$\mathcal{B}(A, T) = \mathcal{B}_l(A, T) \cup \mathcal{B}_r(A, T).$$

A straightforward example

Example (Free monoid on two generators)

- $S = \{a, b\}^*$, $\quad T = \{$words that begin with the letter $a\}$.



Right boundary: $\mathcal{B}_r(\{a, b\}, T) = \{a\}$.

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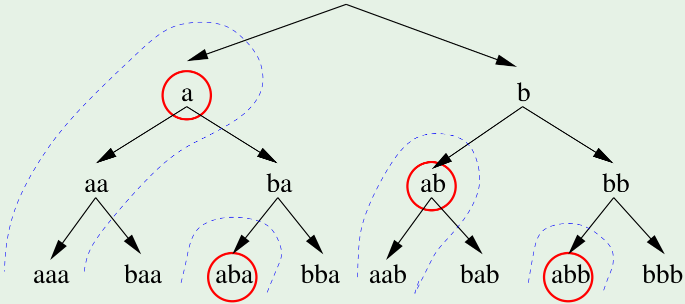

Left boundary: $\mathcal{B}_l(\{a, b\}, T) = \{a\} \cup \{ab\{a, b\}^*\}$.

Basic properties

Proposition (Invariance)

Let A and B be two finite generating sets for a semigroup S. Then $\mathcal{B}_r(A, T)$ is finite if and only if $\mathcal{B}_r(B, T)$ is finite. (The same for left and two-sided.)

Proposition

The following conditions are all sufficient for T to have a finite boundary.

1. $|T| < \infty$
2. $|S \setminus T| < \infty$
3. $S \setminus T$ *is a two-sided ideal of* S

Left and right independence

Example

Let S be the semigroup with set of generators $A = \{a\} \cup B \cup C \cup \{0\}$, where B and C are finite alphabets, and relations R given by:

$$\begin{array}{lll} ab = b, & ba = 0 & b \in B \\ ac = 0, & ca = c & c \in C \\ x0 = 0x = 0 & & x \in A. \end{array}$$

Let $T = \langle A \setminus a \rangle$. Then:

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(i) $N = \{0\} \cup \{a^i : i \in \mathbb{N}\} \cup B \cup C \cup BC \cup CB \cup BCB \cup CBC \cup \ldots$ is a set of normal forms for S;

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(ii) $\mathcal{B}_r(A, T) = \{a^i : i \in \mathbb{N}^0\}A \cap T = B \cup \{0\}$;

(iii) $\mathcal{B}_l(A, T) = A\{a^i : i \in \mathbb{N}^0\} \cap T = C \cup \{0\}$;

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(iii) $\mathcal{B}_l(A, T) = A\{a^i : i \in \mathbb{N}^0\} \cap T = C \cup \{0\}$;

(iv) S and T are both infinite;

(v) $S \setminus T$ is infinite.

Unexpected behavior

Example (Non-transitivity)

Let S be the semigroup (with zero) defined by

$$\langle a, b, c \mid ba = 0, cb = 0, ca = 0 \rangle.$$

Let $T = \langle a, b, bc \rangle$ and $K = \langle a, abc \rangle$. Then $K \leq T \leq S$:

1. K has finite boundary in T
2. T has finite boundary in S
3. K has infinite boundary in S.

Note

This contrasts with finite complement subsemigroups where the property is obviously transitive.

Generators and relations

Proposition

Let $S = \langle A \rangle$ where $|A| < \infty$ and let $T \leq S$. Then T is generated by:

$$X = \mathcal{B}_r(A, T)(S \setminus T)^1 \cap T.$$

Moreover, the generating set X is finite if $\mathcal{B}(A, T)$ is finite.

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Step 1: Let $t \in T$ be arbitrary. Write $t = a_1 a_2 \cdots a_k$ where $a_i \in A$.

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Step 2: Let $\beta_1 = a_1 \cdots a_m$ be the shortest prefix that belongs to T, noting that:

$$\beta_1 = (a_1 \cdots a_{m-1})a_m \in (S \setminus T)A \cap T = \mathcal{B}_r(A, T).$$

$$t = \underbrace{a_1 a_2 \cdots a_m}_{\beta_1} a_{m+1} \cdots a_k$$

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Step 3: If $a_{m+1} \cdots a_k \notin T$ then stop (in this case $t \in X$).

$$t = \beta_1 \underbrace{a_{m+1} \cdots a_k}_{\in T?}$$

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Moreover, the generating set X *is finite if* $\mathcal{B}(A, T)$ *is finite.*

Step 4: Otherwise, let β_2 be the shortest prefix of it that belongs to T. It exists because $a_{m+1} \cdots a_k \in T$. Again $\beta_2 \in \mathcal{B}_r(A, T)$.

$$t = \beta_1 \underbrace{a_{m+1} \cdots a_n}_{\beta_2} a_{n+1} \cdots a_k$$

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Moreover, the generating set X is finite if $\mathcal{B}(A, T)$ is finite.

Eventually: The tail of the word will either be empty or will belong to $S \setminus T$, and we stop.

$$t = \beta_1 \beta_2 \beta_3 \cdots \overbrace{\beta_l \underbrace{a_r \cdots a_k}_{\notin T}}^{\in T}$$

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Corollary

If S is finitely generated and T has a finite boundary in S then T is finitely generated.

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Corollary

If S *is finitely generated and* T *has a finite boundary in* S *then* T *is finitely generated.*

Theorem

If S *is finitely presented and* T *has a finite boundary in* S *then* T *is finitely presented.*

One-sided boundaries

Definition (Bruck–Reilly extension)

M - monoid, $\theta \in \mathrm{End}(M)$. $BR(M,\theta) = \mathbb{N}^0 \times M \times \mathbb{N}^0$ with:

$$(m, a, n)(p, b, q) = (m - n + t, (a\theta^{t-n})(b\theta^{t-p}), q - p + t)$$

where $t = \max(n, p)$.

Proposition

$S = BR(M,\theta)$ finitely generated, and $T = \{(0, a, n) : a \in M, n \in \mathbb{N}^0\}$.

- *T has finite right boundary, T has infinite left boundary.*

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- *T has finite right boundary,* *T has infinite left boundary.*

Example

Choose M and θ such that $BR(M, \theta)$ is finitely presented while M is finitely generated but not finitely presented. Then $T \leq BR(M, \theta)$ is finitely generated and has a finite right boundary, but is not finitely presented.

Strict boundaries and unitary subsemigroups

Problem. If G is an infinite group and H is a proper subgroup of G then $|\mathcal{B}(A, H)| < \infty \Leftrightarrow |H| < \infty$.

Definition (Strict boundary)

$$\mathcal{SB}_r(A, T) = \{t \in T : t = a_1 \cdots a_k \text{ and } a_1 \cdots a_i \notin T \text{ for } 1 \leq i < k\}.$$

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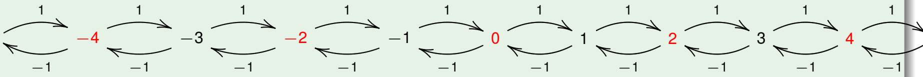
Example

Let $S = \mathbb{Z} = \langle -1, 1 \rangle$ and let $T = 2\mathbb{Z}$. Then:

$$\mathcal{SB}(A, T) = \{0, 2, -2\} \subsetneq 2\mathbb{Z} = \mathcal{B}(A, T).$$



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Theorem

Let S be a finitely presented semigroup with T a subsemigroup of S. If T is left unitary and has a finite strict right boundary in S then T is finitely presented.