Locally-finite connected-homogeneous digraphs

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Symmetry properties for graphs

- There are varying amounts of symmetry that a graph can display.
- Roughly speaking, the more symmetry a graph has the larger its automorphism group will be (and vice versa).

Examples

Γ - graph, $V\Gamma$ - vertex set

- Γ is vertex-transitive if $\operatorname{Aut}\Gamma$ acts transitively on $V\Gamma$.
  - Cayley graphs of groups are vertex transitive.

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Γ - graph, $V\Gamma$ - vertex set

- Γ is vertex-transitive if $\mathrm{Aut}\,\Gamma$ acts transitively on $V\Gamma$.
  - Cayley graphs of groups are vertex transitive.
- Other stronger conditions have been considered:
  - edge-transitive, arc-transitive, k-arc-transitive (Tutte (1947))
  - distance-transitive (Biggs and Smith (1971))
  - homogeneous, k-homogeneous (Fraïssé (1953))
- Concepts like this arise naturally in the theory of permutation groups.

Classification problems

- $\mathcal{P}$ - a symmetry property of graphs

Problem

Classify those graphs Γ satisfying property $\mathcal{P}$.

- Various restrictions can be placed on Γ
  e.g. we may suppose that Γ is:
  - finite
  - infinite but locally-finite
  - countably infinite
  - arbitrary
- In the infinite locally-finite case the number of ends that the graph has plays an important role.

Ends of a graphs

Definition

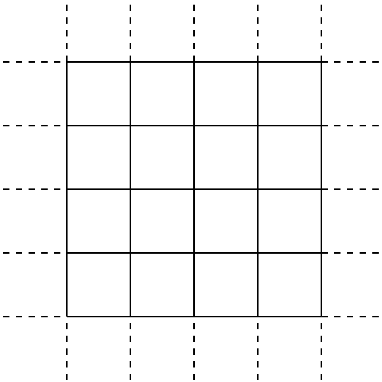
The number of ends of a graph is the least upper bound (possibly ∞) of the number of infinite connected components that can be obtained by removing finitely many edges.

- Intuitively the number of ends corresponds to the number of "ways of going to infinity".

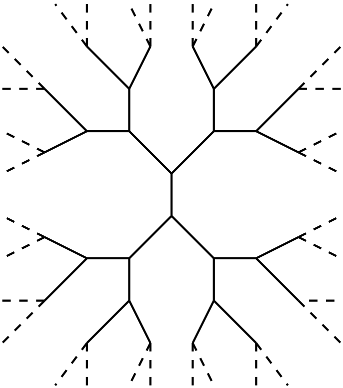
Theorem (Diestel, Jung, Möller (1993))

A connected vertex-transitive graph has either 1, 2 *or* ∞ *many ends.*

Examples: A grid, a tree and a line



Grid has 1 end



Tree has ∞ many ends

Line has 2 ends

Cutting up graphs

Definition (Cuts)

A set $c \subseteq V\Gamma$ of vertices is called a cut if c and its complement c^* are both infinite and

$$\delta c = \{e \in E\Gamma : \text{one vertex of } e \text{ lies in } c \text{ and one in } c^*\}$$

is finite.

Theorem (Dunwoody (1982))

Any infinite connected graph with more than one end has a cut $d \subseteq V\Gamma$ such that for all $g \in \operatorname{Aut}\Gamma$ at least one of the following holds

$$d \subseteq gd, \qquad d \subseteq gd^*, \qquad d^* \subseteq gd, \quad \text{or} \quad d^* \subseteq gd^*.$$

Applications of Dunwoody's theorem

- Dunwoody's theorem has been usefully applied in the study of locally-finite graphs satisfying symmetry conditions.

Examples

- Macpherson (1982) - classification of infinite locally-finite distance-transitive graphs

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Let Γ be a locally finite connected graph with more than one end.

- Möller (1992) - If Γ is 2-distance transitive then Γ is k-distance transitive for all $k \in \mathbb{N}$.
- Thomassen–Woess (1993) - If Γ is 2-arc transitive then Γ is a regular tree.
- Thomassen–Woess (1993) - If Γ is 1-arc transitive and all vertices have degree r, where r is a prime, then Γ is a regular tree.

Digraphs with symmetry

D - a digraph, $ED \subseteq VD \times VD$ - set of arcs of D
(no loops or two-directional arcs $\leftrightarrow$)

Definition

Number of ends of $D :=$ the number of ends of the underlying undirected graph of D.

- Seifter (2007) - investigated the structure of infinite locally-finite transitive digraphs with > 1 end
  - They are far less "sensitive" to symmetry conditions than undirected graphs.
  - Even with a seemingly very strong condition called high-arc-transitivity they can have very rich structure.

Connected-homogeneity

Definition

A digraph D is called connected-homogeneous if any isomorphism between finite connected induced subdigraphs of D extends to an automorphism.

Example. $D =$ infinite directed line (i.e. $\mathbb{Z}$ with arcs $i \to i + 1$)

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Problem. Classify the countable connected-homogeneous digraphs.

A solution to this problem would complete the following table:

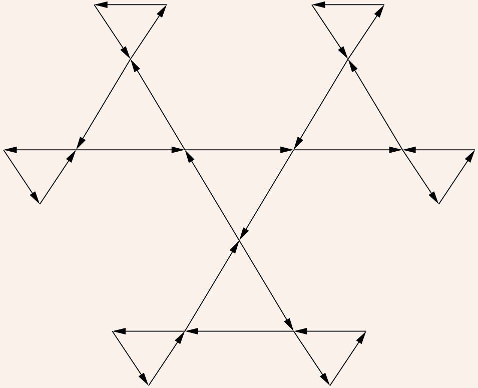
	Homogeneous	Connected-homogeneous
Graphs	Gardiner (1976), Lachlan & Woodrow (1980)	Gardiner (1978), Enomoto (1981) RG & Macpherson (2007)
Digraphs	Lachlan (1982), Cherlin (1998)	**?**

Subproblem. Classify the connected-homogeneous digraphs that are locally-finite and have more than one end.

The case that a triangle embeds

Theorem (RG & Möller (2008))

Let D be a connected locally-finite digraph with more than one end, and suppose that D embeds a triangle.
Then D is connected-homogeneous if and only if it is isomorphic to a digraph built from directed triangles in the following way:



Highly arc-transitive digraphs

Definition

A k-arc in D is a sequence $(x_0, \ldots, x_k)$ of vertices with $x_i \to x_{i+1}$ (and $x_{i-1} \neq x_{i+1}$).

A digraph D is highly-arc-transitive if $\operatorname{Aut} D$ is transitive on the set of k-arcs of D for every natural number k.

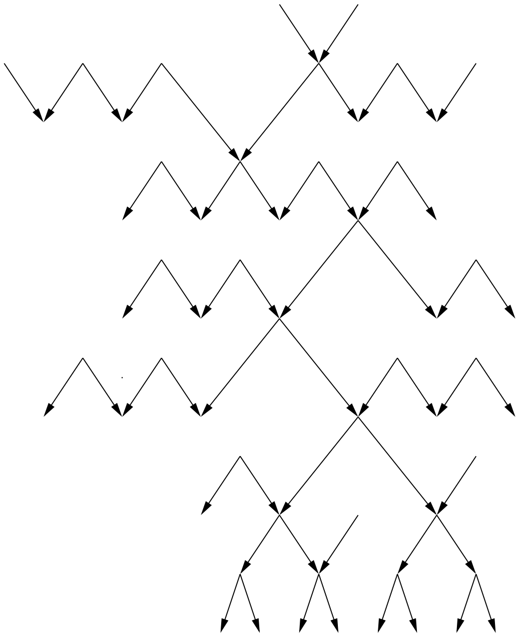
- Cameron, Praeger, and Wormald (1993) - carried out an extensive study of the class of highly-arc-transitive digraphs.

Proposition (RG & Möller (2008))

Let D be a triangle-free locally-finite digraph with more than one end. If D is connected-homogeneous then D is highly-arc-transitive.

Triangle-free connected-homogeneous digraphs

Directed regular trees



Triangle-free connected-homogeneous digraphs

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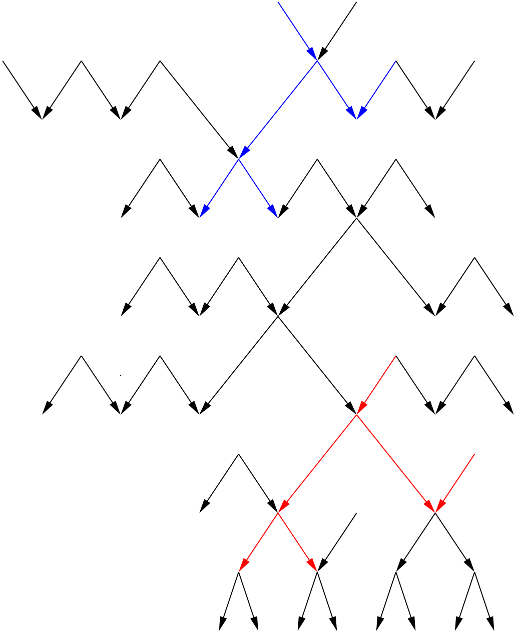
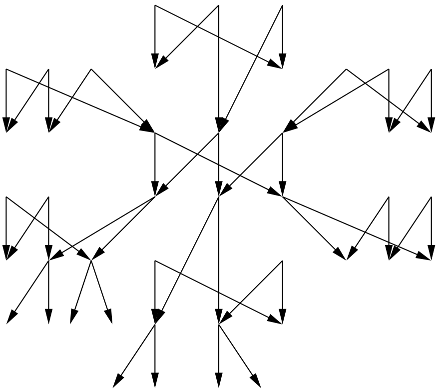


Triangle-free connected-homogeneous digraphs

Other tree-like examples exist.

Constructed by gluing together certain bipartite graphs.

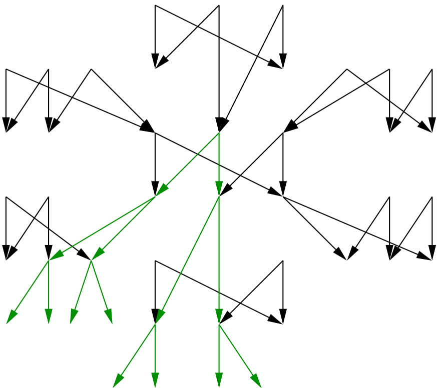


Triangle-free connected-homogeneous digraphs

Definition

The set of descendants $\mathrm{desc}(u)$ of a vertex u is the set of all vertices v such that there is a directed path from u to v.

In this example $\mathrm{desc}(u)$ is a tree for every vertex u.

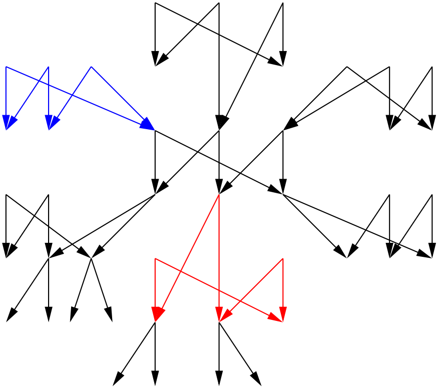


Triangle-free connected-homogeneous digraphs

Definition

The reachability digraph $\Delta(D)$ of D is the subdigraph induced by the set of all arcs reachable by an alternating walk beginning from an arc.

In this example $\Delta(D)$ is bipartite and is isomorphic to a 6-cycle.



Triangle-free case

- For arbitrary locally finite highly-arc-transitive digraphs
  - $\mathrm{desc}(u)$ need not be a tree
  - it is an open question as to whether $\Delta(D)$ is bipartite

Theorem (RG & Möller (2008))

Let D be a triangle-free locally-finite connected-homogeneous digraph with infinitely many ends. Then

- $\mathrm{desc}(u)$ is a tree for all $u \in VD$
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- $\mathrm{desc}(u)$ is a tree for all $u \in VD$
- $\Delta(D)$ is a bipartite graph
- Specifically, $\Delta(D)$ is isomorphic to one of:
  - infinite semiregular tree $T_{a,b}$ $(a, b \in \mathbb{N})$
  - cycle C_{2m} $(m \geq 4)$
  - complete bipartite graph $K_{m,n}$ $(m, n \in \mathbb{N}$ with $m \geq 2$ or $n \geq 2)$
  - complement of a perfect matching CP_n for some $n \geq 3$ (i.e. the complete bipartite graph $K_{n,n}$ with a matching removed)

- **Proof.** Uses Dunwoody's theorem, structure trees, and results from Seifter (2007).

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- But do all of these potential reachability graphs actually arise in examples?

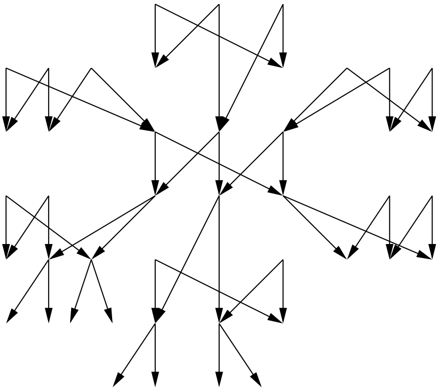
CPW's universal covering construction.

- Let Δ be one of the following:

semiregular tree	complete bipartite
complement of perfect matching	cycle

- Then there exists a connected-homogeneous digraph $DL(\Delta)$ with reachability graph Δ.
- $DL(\Delta)$ is constructed by gluing together copies of Δ in such a way that
  - any two copies of Δ intersect in at most one vertex
  - the only cycles in D are those that occur in the copies of Δ
- This construction was introduced by Cameron, Praeger, and Wormald (1993) during their study of highly-arc-transitive digraphs.

CPW's universal covering construction example



- The digraph $DL(\Delta)$ where $\Delta = C_6$ is a 6-cycle.

Triangle-free case

- And "most" examples actually arise in this way.

Theorem (RG & Möller (2008))

Let D be a connected triangle-free locally-finite connected-homogeneous digraph with infinitely many ends, and with $\Delta(D)$ *not isomorphic to* $K_{2,2}$ *or to the complement of a perfect matching.*

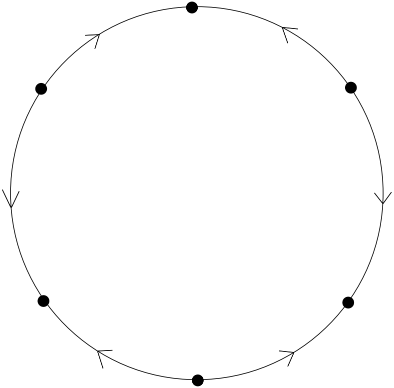
Then $D \cong DL(\Delta)$*, the digraph obtained from the above CPW universal covering construction.*

In particular, in these cases D is uniquely determined by its reachability digraph $\Delta(D)$*.*

- This just leaves the cases that Δ is isomorphic to $K_{2,2}$ or to the complement of a perfect matching.

$\Delta \cong$ complement of perfect matching

- Malnič, Marušič, Seifter, and Zgrablić (2002)
  - introduced a new family of highly-arc-transitive digraphs
  - answered an open question about homomorphisms onto Z
- The original construction involved gluing together cycles $C_{2m} (m \geq 3)$.



An MMSZ digraph D with $\Delta(D) \cong CP_3$ complement of perfect matching.

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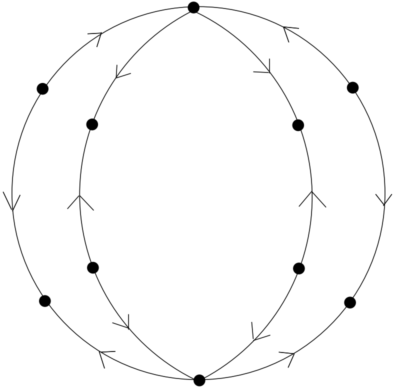

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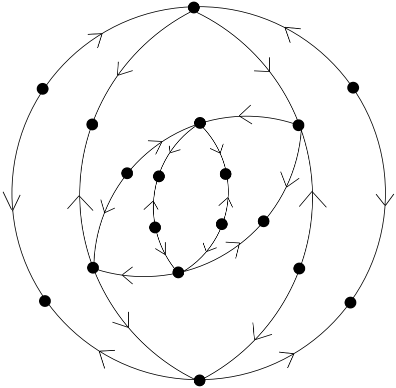

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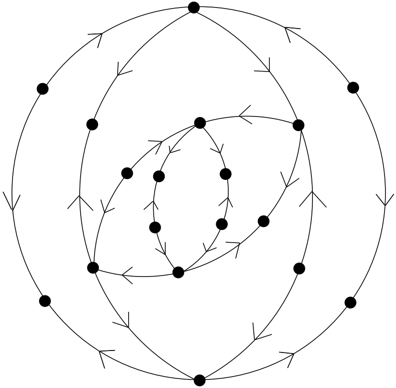

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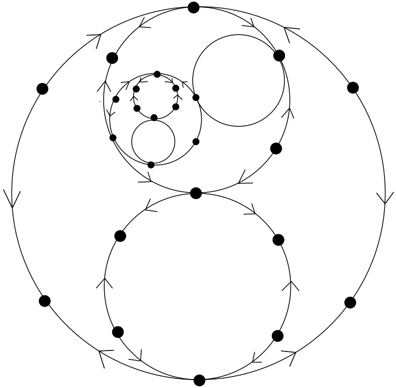
- Carrying out their construction with any complement of perfect matching $CP_m (m \geq 3)$ gives a connected-homogeneous digraph.
- A generalisation of their construction gives further examples.



A generalised MMSZ digraph

$\Delta \cong$ complement of perfect matching

Theorem (RG & Möller (2008))

Let D be a connected triangle-free locally-finite connected-homogeneous digraph with more than one end.

If $\Delta(D)$ is isomorphic to the complement of a perfect matching then either

(i) *D is obtained from the CPW construction or*

(ii) *D is a generalised MMSZ digraph*

- In particular, for any complement of perfect matching Δ there are infinitely many non-isomorphic D with $\Delta(D) \cong \Delta$.

$\Delta \cong K_{2,2}$ - the problem case

D - connected triangle-free locally-finite connected-homogeneous digraph with more than one end.

Suppose $\Delta(D) \cong K_{2,2}$

- Known examples
  - CPW example, and
  - generalised MMSZ examples
- But there are other examples in addition to these (too complicated to go into here :-( ).
- This is the only case where the infinitely-ended classification is still incomplete.

Concluding remarks

- We have results for the 2-ended case, e.g. $\Delta(D) \cong K_{n,n}$.

Still to do

- Complete the classification by determining all examples whose reachability graph is $K_{2,2}$.

And then

- Extend the result to:
  - non-locally finite digraphs
  - one-ended digraphs
- Generalise results to locally-finite highly-arc-transitive digraphs with more than one end.
  - In particular prove that for such digraphs $\Delta(D)$ is always bipartite.