Investigating groups of units of special monoids using boundaries in Schützenberger graphs

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Monoid and group presentations

$$\mathrm{Mon}\langle A \mid R\rangle = \langle \underbrace{a_1, \ldots, a_n}_{\text{letters / generators}} \mid \underbrace{u_1 = v_1, \ldots, u_m = v_m}_{\text{words / defining relations}} \rangle$$

Example: $M \cong \mathrm{Mon}\langle A \mid R\rangle = \mathrm{Mon}\langle a, b \mid ab = ba\rangle$

Words $u, v \in A^*$ represent the same element of M if u can be transformed into v by a finite number of applications of the relations.

$$\text{e.g.} \quad abaa = aaba = aaab, \qquad abb \neq aab.$$

Here every word is equal to to a unique word of the form $a^i b^j$.

$$\mathrm{Gp}\langle A \mid R\rangle = \mathrm{Mon}\langle A \cup A^{-1} \mid R, aa^{-1} = 1, a^{-1}a = 1 (a \in A)\rangle$$

Example: The free group

$\mathrm{FG}(A) = \mathrm{Gp}\langle A \mid \ \rangle = \mathrm{Mon}\langle A \cup A^{-1} \mid aa^{-1} = 1,\ a^{-1}a = 1\ (a \in A)\rangle$

The word problem for semigroups and groups

Definition

A monoid S with a finite generating set A has decidable word problem if there is an algorithm which for any two words $w_1, w_2 \in A^*$ decides whether or not they represent the same element of S.

Example. $\mathrm{Mon}\langle a, b \mid ab = ba \rangle$ has decidable word problem.

Some history

- Markov (1947) and Post (1947): first examples of finitely presented semigroups with undecidable word problem;
- Novikov (1955) and Boone (1958): finitely presented group with undecidable word problem.

Longstanding open problem

Is the word problem decidable for one-relation monoids $\mathrm{Mon}\langle A \mid u = v \rangle$?

One relator groups and monoids

Magnus (1932): Proved that the word problem is decidable for one-relator groups $\mathrm{Gp}\langle A \mid r = 1\rangle$.

- Magnus's "break-down procedure" uses Reidemeister–Schreier rewriting, free products with amalgamation, and HHN extensions.
- Proof is by induction on the length of the relator e.g. in some cases $\mathrm{Gp}\langle A \mid r = 1\rangle$ is an HNN extension of a one-relator group with shorter defining relation.

Adjan (1966): For one-relation monoids proved:

- $\mathrm{Mon}\langle A \mid u = 1\rangle$ has decidable word problem.
- $\mathrm{Mon}\langle A \mid u = v\rangle$ has decidable word problem if $u, v \in A^*$ are both non-empty and have different initial and different terminal letters.

Other work by Lallement (1974), Squier and Wrathall (1983), Zhang (1991), Adjan and Oganessian (1987), Kobayashi (2000).

Zhang's theory of special monoid presentations

A special monoid presentation is one of the form

$$\mathrm{Mon}\langle A \mid w_1 = 1, w_2 = 1, \ldots, w_k = 1 \rangle.$$

Let M be the monoid defined by the above presentation.

Theorem (Zhang (1992))

The group of units G of M admits a finite presentation

$$\mathrm{Gp}\langle B \mid \beta_1 = 1, \beta_2 = 1, \ldots, \beta_k = 1 \rangle$$

with the same number of defining relators as in the presentation for M.

- There is an algorithm which computes this presentation for G.
- M has decidable word problem $\Leftrightarrow$ G has decidable word problem.

Corollary: The group of units of $\mathrm{Mon}\langle A \mid u = 1 \rangle$ is a one-relator group and hence by Zhang + Magnus $\mathrm{Mon}\langle A \mid u = 1 \rangle$ has decidable word problem.

Inverse monoid presentations

An inverse monoid is a monoid M such that for every $x \in M$ there is a unique $x^{-1} \in M$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

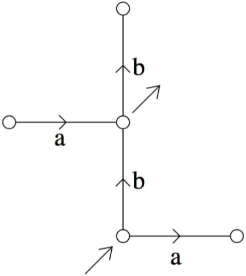
For all $x, y \in M$ we have

$$x = xx^{-1}x,\ (x^{-1})^{-1} = x,\ (xy)^{-1} = y^{-1}x^{-1},\ xx^{-1}yy^{-1} = yy^{-1}xx^{-1} \quad (\dagger)$$

$$\mathrm{Inv}\langle A \mid R \rangle = \mathrm{Mon}\langle A \cup A^{-1} \mid R \cup (\dagger) \rangle$$

where x, y range over all possible words from $(A \cup A^{-1})^*$.

Free inverse monoid $\mathrm{FIM}(A) = \mathrm{Inv}\langle A \mid \ \rangle$



Munn (1974)
Elements of $\mathrm{FIM}(A)$ can be represented using Munn trees. e.g. in $\mathrm{FIM}(a, b)$ we have $u = w$ where

$u = aa^{-1}bb^{-1}ba^{-1}abb^{-1}$
$w = bbb^{-1}a^{-1}ab^{-1}aa^{-1}b$

One-relator inverse monoids

Open problem

Is the word problem decidable for special one-relator inverse monoids $\mathrm{Inv}\langle A \mid w = 1\rangle$?

This is important because...

Theorem (Ivanov, Margolis, Meakin (2001))

If the word problem is decidable for all inverse monoids of the form $\mathrm{Inv}\langle A \mid w = 1\rangle$ then the word problem is also decidable for every one-relator monoid $\mathrm{Mon}\langle A \mid u = v\rangle$.

Word problem for $\mathrm{Inv}\langle A \mid w = 1\rangle$ is known to be decidable in some cases:

- idempotent relator Birget, Margolis, Meakin (1994)
- 'strictly positive' type Ivanov, Margolis, Meakin (2001)
- 'Adian type' / 'Baumslag-Solitar type' Margolis, Meakin, Sunik (2005)
- sparse relator Hermiller, Lindblad, Meakin (2009)

Special inverse monoid presentations

A special inverse monoid presentation is one of the form

$$\mathrm{Inv}\langle A \mid w_1 = 1, w_2 = 1, \ldots, w_k = 1\rangle.$$

Let M be the monoid defined by the above presentation.

Idea: Develop a Zhang-style theory for special inverse monoids

- Is the group of units G of M finitely presented?
  - (G is known to be finitely generated. Stephen's procedure for Schützenberger graphs $\Rightarrow$ G is generated by the set of invertible prefixes of the w_i (Ivanov, Margolis, Meakin, 2001))
- If so, is there an algorithm which computes a finite presentation for G?
- Is it true that M has decidable word problem $\Leftrightarrow$ G has decidable word problem?

Problem: Is the group of units of $\mathrm{Inv}\langle A \mid w = 1\rangle$ a one-relator group?

Cayley graphs and Schützenberger graphs

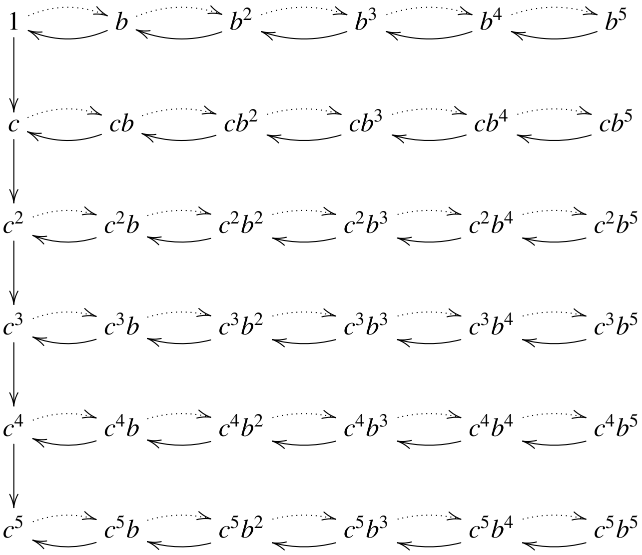
M - monoid generated by a finite set A.

The (right) Cayley graph $\Gamma(M, A)$
Vertices: M
Directed edges: $x \xrightarrow{a} y$ iff $y = xa$ where $x, y \in M$, $a \in A$.

Directed distance: $d_A(x, y) =$ the minimum length of a word $a_1 a_2 \cdots a_r \in A^*$ with the property that $x a_1 a_2 \cdots a_r = y$, or ∞ if there is no such word.

Schützenberger graphs: Given an $\mathcal{R}$-class R of M, the Schützenberger graph $\Gamma(R)$ of R is the subgraph of the Cayley graph induced on R. These are the strongly connected components of Cayley graph.

Cayley graphs of semigroups and monoids



The bicyclic monoid $B = \langle b, c \mid bc = 1 \rangle$

Boundaries

Definition

Let X be a set of vertices in a digraph Γ. We call $(x, y) \in X \times X$ a boundary pair of X if there is a path $e_1 e_2 \dots e_m$ with the following properties: $\iota e_1 = x$, $\tau e_m = y$, and $\iota e_2, \iota e_3, \dots, \iota e_m$ all belong to $V \setminus X$. Define

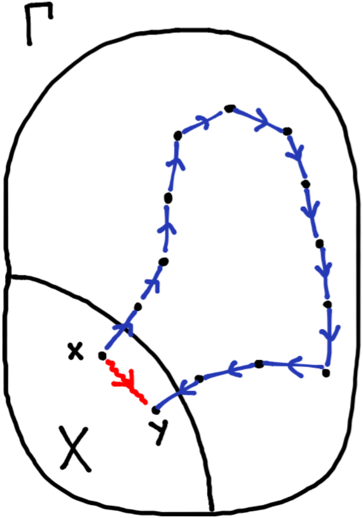
$$\beta(X) = \sup\{d(x, y) \ : \ (x, y) \text{ is a boundary pair}\}$$

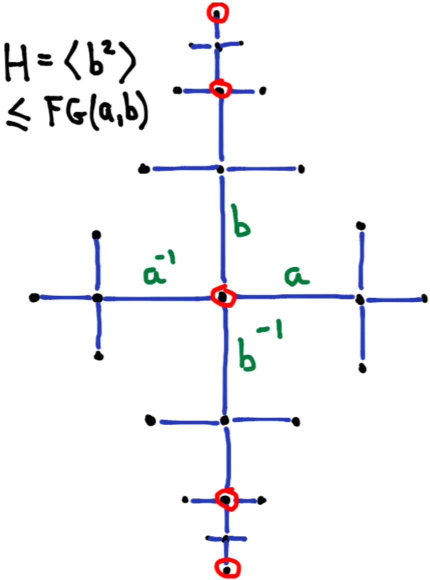
where $d(x, y)$ is the directed distance from x to y. We say that X has a finite boundary in Γ if $\beta(X)$ is finite.

Definition

Let M be a monoid generated by a finite set A. We say $X \subseteq M$ has a finite right boundary (with respect to A) if X has finite boundary inside the right Cayley graph $\Gamma(M, A)$.

Finite boundaries in digraphs and monoids





Boundary theory (I)

A subsemigroup $T \leq S$ is left unitary if for all $s \in S$, $t \in T$ we have $ts \in T \Rightarrow s \in T$.

Theorem (RDG (2006))

Let S be a finitely generated monoid and let T be a submonoid of S. If T is left unitary and has a finite right boundary then T is finitely generated. Moreover, if S is finitely presented then T is finitely presented.

Theorem (RDG & Ruskuc (2016))

Let M be a finitely presented special monoid

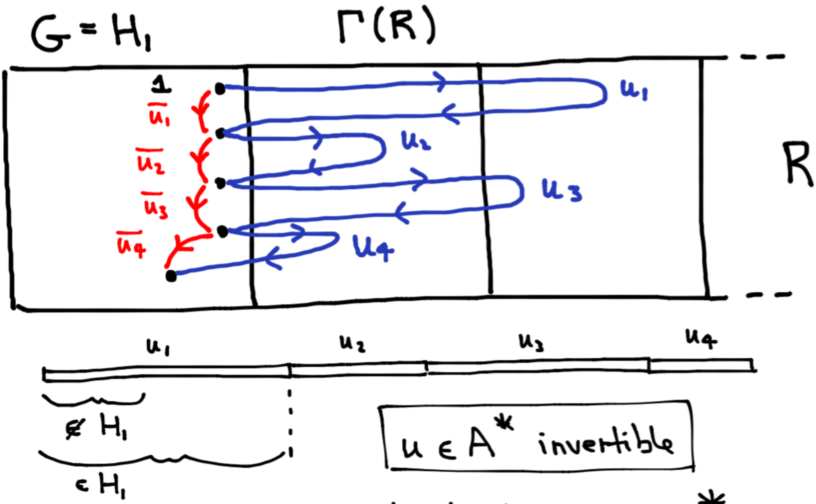
$$M \cong \mathrm{Mon}\langle A \mid w_1 = 1, w_2 = 1, \ldots, w_k = 1 \rangle.$$

Then the group of units G of M has a finite right boundary and so is finitely presented. Moreover, G admits a finite presenation of the form

$$\mathrm{Gp}\langle B \mid \phi(w_1) = 1, \phi(w_2) = 1, \ldots, \phi(w_k) = 1 \rangle$$

where ϕ is the RS-rewriting mapping in the above theorem.

Boundary of the units in special monoids



$$\phi(u) = \phi(u_1 u_2 u_3 u_4) = b_{\overline{u_1}}\, b_{\overline{u_2}}\, b_{\overline{u_3}}\, b_{\overline{u_4}} \in B^*$$

Boundaries in special inverse monoids

Does this generalise to special inverse monoids?

Proposition (RDG & Ruskuc (2016))

Let M be the inverse monoid defined by the presentation:

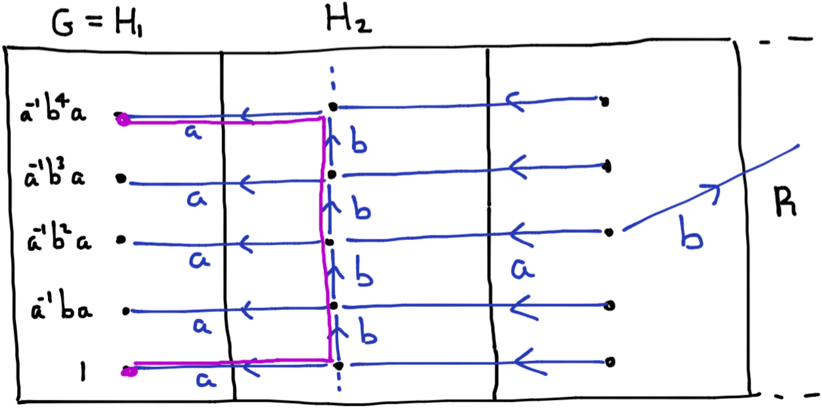
$$\begin{aligned} & \mathrm{Inv}\langle a, b \mid (a^{-1}ba)(a^{-1}ba)^{-1}(a^{-1}ba)^{-1}(a^{-1}ba) = 1 \rangle \\ \cong \quad & \mathrm{Inv}\langle a, b \mid (a^{-1}ba)(a^{-1}ba)^{-1} = 1,\ (a^{-1}ba)^{-1}(a^{-1}ba) = 1 \rangle. \end{aligned}$$

The group of units G of M has an infinite right boundary in M.

The basic idea

This presentation says M is an inverse monoid generated by a, b such that the element represented by $(a^{-1}ba)$ is invertible.

Boundaries in special inverse monoids

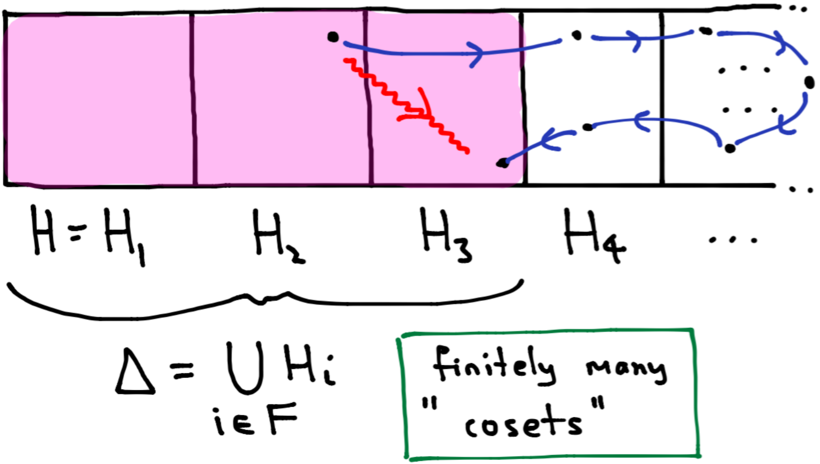


$G = H_1$ has ∞ boundary. $H_1 \cup H_2$ has finite boundary

<u>Note</u> : $\Gamma(R)$ is a tree in this example.

Boundary theory (II)

M – f.g. monoid, H – a maximal subgroup



Finite union of $\mathcal{H}$-classes with finite boundary

Let M be a monoid generated by a finite set A.

Definition

Let H be a maximal subgroup of M. We say H has a finite cover with finite right boundary if there is a finite set of $\mathcal{H}$-classes $\Delta = \cup_{i \in F} H_i$, in the $\mathcal{R}$-class R of H, with $H \subseteq \Delta$ such that Δ has a finite right boundary.

Theorem (RDG and Ruskuc (2016))

Let H be a maximal subgroup of M that has a finite cover with finite right boundary. Then H is finitely generated. Moreover, if M is finitely presented then H is finitely presented.

Applications: Can to prove finite presentability of the group of units of $\mathrm{Inv}\langle A \mid w = 1 \rangle$ in certain cases:

- e.g. idempotent relator, sparse relator, when the Schützenberger graph is sufficiently tree-like (finite tree width).

Question: Does the group of units of $\mathrm{Inv}\langle A \mid w = 1 \rangle$ always have a finite cover with finite boundary?

Quasi-isometries

Theorem (Folklore?)

Let G be a group generated by a finite set A and let H be a subgroup of G. Then H is finitely generated if and only if there is a finite collection of right cosets Hg_j $(j \in J)$ such that the subgraph Δ of the Cayley graph $\Gamma(G, A)$ induced on the set $\cup_{j \in J} Hg_j$ is connected. Moreover, in this case, H is quasi-isometric to the graph Δ.

Theorem (RDG and Ruskuc (2016))

Let H be a maximal subgroup of a finitely generated monoid M. Then H is finitely generated if and only if there is a finite collection of $\mathcal{H}$-classes $H_j (j \in J)$ in the $\mathcal{R}$-class R of H such that the subgraph Δ of the Schützenberger graph $\Gamma(R)$ induced on the set $\cup_{j \in J} H_j$ is strongly connected. Moreover, in this case, H is quasi-isometric to Δ.

Conclusion: Whether or not H is finitely presented can be "seen" in the geometry of the subgraph Δ of the Schützenberger graph $\Gamma(R)$.

Coherence and positive relators

A group G is coherent if ever finitely generated subgroup of G is finitely presented.

Proposition (RDG and Ruskuc (2016))

Let M be the inverse monoid defined by a special one-relator presentation $\mathrm{Inv}\langle A \mid w = 1 \rangle$ where w is a cyclically reduced word. If $\mathrm{Gp}\langle A \mid w = 1 \rangle$ is coherent then the group of units of M is finitely presented.

Conjecture (Gilbert Baumslag (1974))

Every one-relator group is coherent.

"Positive one-relator groups are coherent" D. T. Wise (2003) (not yet published since relies on another paper of Wise for which there is still a gap in one of the proofs).

This all strongly suggests that if $w \in A^+$ (i.e. is a strictly positive word) then the group of units of $\mathrm{Inv}\langle A \mid w = 1 \rangle$ should be finitely presented.

Margolis–Meakin O'Hare monoid

$$M \cong \mathrm{Inv}\langle a, b, c, d \mid \underbrace{abcd}_{\gamma}\,\underbrace{acd}_{\beta}\,\underbrace{ad}_{\alpha}\,\underbrace{abbcd}_{\delta}\,\underbrace{acd}_{\beta} = 1\rangle.$$

Margolis and Meakin showed that $ad, acd, abcd, abbcd$ are all invertible.

Proposition (RDG and Ruskuc (2016))

The group of units G of M is the one-relator group

$$\mathrm{Gp}\langle b, c, y \mid bcycyybbcycy = 1\rangle.$$

Proof ideas:

$$\mathrm{Inv}\langle A \mid abcdacdadabbcdacd = 1\rangle =$$
$$\mathrm{Inv}\langle A \mid \underbrace{aba^{-1}}\,\underbrace{aca^{-1}}\,\underbrace{ad}\,\underbrace{aca^{-1}}\,\underbrace{ad}\,\underbrace{ad}\,\underbrace{aba^{-1}}\,\underbrace{aba^{-1}}\,\underbrace{aca^{-1}}\,\underbrace{ad}\,\underbrace{aca^{-1}}\,\underbrace{ad} = 1\rangle$$

Where $\{ad, aca^{-1}, aba^{-1}\}$ is a free generating set for the subgroup of $\mathrm{FG}(a, b, c, d)$ generated by $\{ad, acd, abcd, abbcd\}$.