Finite semigroups and their generating sets

Robert Gray

University of Leeds

May 2006

Outline

1. Some semigroup theory
   - Basic definitions, examples, generating sets
   - Green's relations and the Rees theorem
2. Generating sets for completely 0-simple semigroups
   - 0-simple semigroups and their associated graphs
   - Main results
3. Applications
   - Brandt semigroups and I_n
   - Subsemigroups of T_n

Definition and examples

Definition

A *semigroup* is a pair $S = (S, \cdot)$ where S is a set and $\cdot$ is a binary operation satisfying the associative law.

Examples

- Groups (are semigroups that satisfy $(\forall a \in S)\ aS = S \ \&\ Sa = S$).
- Subsemigroups of groups (e.g. $(\mathbb{N}, +) \leq (\mathbb{Z}, +)$).

Definition and examples

Definition

A *semigroup* is a pair $S = (S, \cdot)$ where S is a set and $\cdot$ is a binary operation satisfying the associative law.

Examples

- Groups (are semigroups that satisfy $(\forall a \in S)\ aS = S \ \&\ Sa = S$).
- Subsemigroups of groups (e.g. $(\mathbb{N}, +) \leq (\mathbb{Z}, +)$).
- Right and left zero semigroups

$$(\forall x, y \in S)\ xy = y, \qquad (\forall x, y \in S)\ xy = x.$$

- Rectangular bands

$$(\forall x, y \in S)\ xyx = x.$$

  Every rectangular band is isomorphic to a semigroup $A \times B$, where A and B are non-empty sets, with multiplication:

$$(a_1, b_1)(a_2, b_2) = (a_1, b_2).$$

Semigroups of transformations

Let X be a set. The *full transformation semigroup* T_X is the semigroup of all maps from X to X under composition. When $|X| = n$ we write $T_X = T_n$.

- P_X (partial transformation semigroup): The semigroup of all *partial maps* of X. For example in P_4:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & - & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & 4 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & - & - \end{pmatrix}.$$

- I_X (symmetric inverse semigroup): The semigroup of all *partial one-one maps* of X. For example in I_4:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & - & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & - & - \end{pmatrix}.$$

Endomorphism monoids

Given a mathematical structure M the set of endomorphisms of M (denoted $\mathrm{End}(M)$) forms a monoid (semigroup with identity).

- When $M = X$ is simply a set (no structure) we have $\mathrm{End}(M) \cong T_X$ the full transformation semigroup.

Endomorphism monoids

Given a mathematical structure M the set of endomorphisms of M (denoted $\mathrm{End}(M)$) forms a monoid (semigroup with identity).

- When $M = X$ is simply a set (no structure) we have $\mathrm{End}(M) \cong T_X$ the full transformation semigroup.
- When $M = V$ an n-dimensional vector space over $GF(q)$ then $\mathrm{End}(M) \cong \mathrm{GLS}(n, q)$ the *general linear semigroup* of *all* $n \times n$ matrices over $GF(q)$.
- When $M = Y_n = (\{1, \ldots, n\}, \leq)$ (an n-element chain) then $\mathrm{End}(M)$ is isomorphic to the semigroup of order preserving transformations:

$$O_n = \{\alpha \in T_n : (\forall x, y \in X_n)\ x \leq y \Rightarrow x\alpha \leq y\alpha\}.$$

Generating sets

Definition

Given $A \subseteq S$ let $\langle A \rangle$ denote the subsemigroup of S generated by the set A.

Question. Given a semigroup S how many elements do we need in order to generate S?

Definition

We use $\mathrm{rank}(S)$ to denote the minimum cardinality of a generating set for a semigroup S:

$$\mathrm{rank}(S) = \min\{|A| : A \subseteq S \ \& \ \langle A \rangle = S\}.$$

Small generating sets

Example

Groups

- $\text{rank}(\mathbb{Z}_n) = 1, \text{rank}(S_n) = 2$, $\text{rank}(A_n) = 2$, $\text{rank}(D_{2n}) = 2$, $\text{rank}(Q_8) = 2$.

Semigroups

- k element left (right) zero semigroups have rank k.
- $\text{rank}(T_n) = 3$ (a transposition, n-cycle and one transformation α with $|\text{im}(\alpha)| = n - 1$).
- $\text{rank}(P_n) = 4$ (partial transformations).
- $\text{rank}(I_n) = 3$ (symmetric inverse semigroup).
- $\text{rank}(\text{GLS}(n, q)) = 3$ (general linear semigroup).

Ideals and singular mappings

The ranks of the proper two-sided ideals of each of the above semigroups of transformations have been considered.

Semigroup	**Author(s)**
Full transformation semigroup	Gomes & Howie (1987) Howie & McFadden (1990)
Partial transformation semigroup	Garba (1994)
Symmetric inverse semigroup	Garba (1994) Gomes & Howie (1987)
Order preserving transformations	Gomes & Howie (1992) Yang (1998)
General linear semigroup	Dawlings (1982)

The main problem

The “building blocks” of arbitrary finite semigroups are the (so called) *completely* 0*-simple semigroups*.

For each of the examples above the problem reduces to that of determining the rank of a corresponding completely 0-simple semigroup.

Question. Can we find an expression for the rank of an arbitrary finite completely 0-simple semigroup?

The main problem

The “building blocks” of arbitrary finite semigroups are the (so called) *completely* 0*-simple semigroups*.

For each of the examples above the problem reduces to that of determining the rank of a corresponding completely 0-simple semigroup.

Question. Can we find an expression for the rank of an arbitrary finite completely 0-simple semigroup?

Such a result could be regarded as an analogue in semigroup theory of the following result for groups.

Proposition

Every non-abelian finite simple group G satisfies $\mathrm{rank}(G) = 2$.

Proof. Follows from the classification of finite simple groups. □

Completely 0-simple semigroups

Definition

Let T be a subsemigroup of S.

- If $ST \subseteq T$ then T is called a *left ideal*.
- If $TS \subseteq T$ then T is called a *right ideal*.
- T is called a (two-sided) ideal if it is both a left and a right ideal.

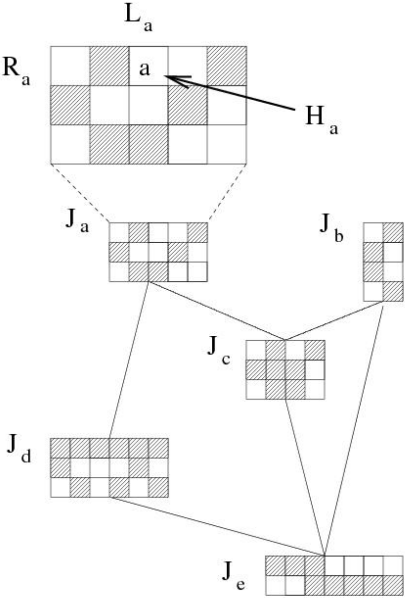
Definition

A semigroup is called *simple* if it has no proper ideals.

A semigroup S with 0 is called 0*-simple* if $\{0\}$ and S are its only ideals, and is called *completely* 0*-simple* if it is 0-simple and is *group bound*.

Fact. Every finite 0-simple semigroup is completely 0-simple.

Green's relations

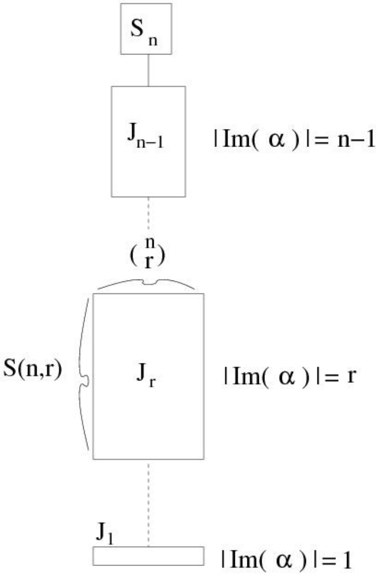


S - semigroup, $x, y \in S$

$$x\mathcal{R}y \Leftrightarrow xS^1 = yS^1$$
$$x\mathcal{L}y \Leftrightarrow S^1x = S^1y$$
$$x\mathcal{J}y \Leftrightarrow S^1xS^1 = S^1yS^1$$

- $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R} (= \mathcal{J})$
- $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$
- $J_x \leq J_y \Leftrightarrow S^1xS^1 \subseteq S^1yS^1$

Green's relations in T_n



T_n - full transformation semigroup, $\alpha, \beta \in T_n$

$$\begin{aligned} \alpha \mathcal{L} \beta &\Leftrightarrow \text{im}(\alpha) = \text{im}(\beta) \\ \alpha \mathcal{R} \beta &\Leftrightarrow \ker(\alpha) = \ker(\beta) \\ \alpha \mathcal{J} \beta &\Leftrightarrow |\text{im}(\alpha)| = |\text{im}(\beta)| \end{aligned}$$

$$J_r = \{\alpha \in T_n : |\text{im}(\alpha)| = r\}$$

The ideals of T_n:

$$\begin{aligned} K(n,r) &= \{\alpha \in T_n : |\text{im}(\alpha)| \leq r\} \\ &= J_r \cup \cdots \cup J_1. \end{aligned}$$

Example: the semigroup $\mathrm{Sing}_3 = T_3 \setminus S_3$

	$\{1,2\}$	$\{2,3\}$	$\{1,3\}$
$\{3\},\{1,2\}$	$[1,1,2],[2,2,1]$	$[2,2,3],[3,3,2]$	$[1,1,3],[3,3,1]$
$\{1\},\{2,3\}$	$[1,2,2],[2,1,1]$	$[2,3,3],[3,2,2]$	$[1,3,3],[3,1,1]$
$\{2\},\{1,3\}$	$[1,2,1],[2,1,2]$	$[2,3,2],[3,2,3]$	$[1,3,1],[3,1,3]$

	$\{1\}$	$\{2\}$	$\{3\}$
$\{1,2,3\}$	$[1,1,1]$	$[2,2,2]$	$[3,3,3]$

In this case the dimensions of the maximal $\mathcal{J}$-class are given by:

$$S(3,2) = 3, \quad \& \quad \binom{3}{2} = 3.$$

Principal factors

Definition (Principal factor)

Let J be some $\mathcal{J}$-class of a semigroup S. Then the principal factor of S corresponding to J is the set $J^* = J \cup \{0\}$ with multiplication

$$s * t = \begin{cases} st & \text{if } s, t, st \in J \\ 0 & \text{otherwise.} \end{cases}$$

Fact. The semigroup J^* is either a semigroup with zero multiplication or is a 0-simple semigroup.

Rees matrix semigroups

Definition

- G - a group; I, Λ - non-empty index sets.
- $P = (p_{\lambda i})$ a *regular* $|\Lambda| \times |I|$ matrix over $G \cup \{0\}$.
- $S = (I \times G \times \Lambda) \cup \{0\}$ with multiplication

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, gp_{\lambda j}h, \mu) & : \quad p_{\lambda j} \neq 0 \\ 0 & : \quad \text{otherwise} \end{cases}$$

$$(i, g, \lambda)0 = 0(i, g, \lambda) = 00 = 0.$$

Theorem (The Rees Theorem)

A semigroup S is completely 0-simple if and only if it is isomorphic to $\mathcal{M}^0[G; I, \Lambda; P]$ where G is a group and P is regular.

Rees matrix semigroups

The original problem:

Problem. Find an expression for the rank of an arbitrary finite completely 0-simple semigroup.

As a consequence of the Rees Theorem this reduces to:

Problem. Find a formula (in terms of G, I, Λ and P) for the rank of an arbitrary finite Rees matrix semigroup.

A special case

A semigroup is called *combinatorial* if its maximal subgroups are all trivial. The combinatorial completely 0-simple semigroups are called *rectangular* 0*-bands*.

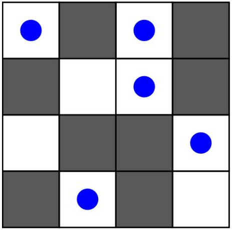
Definition (Rectangular 0-band)

- $I = \{1, \ldots, m\}, \Lambda = \{1, \ldots, n\}$
- $P = (p_{\lambda i})$ a *regular* $\Lambda \times I$ matrix over $\{0, 1\}$.
- $S = (I \times \Lambda) \cup \{0\}$ with multiplication

$$(i, \lambda)(j, \mu) = \begin{cases} (i, \mu) & : \quad p_{\lambda j} = 1 \\ 0 & : \quad \text{otherwise} \end{cases}, \quad (i, \lambda)0 = 0(i, \lambda) = 00 = 0.$$

Associated with every $S = \mathcal{M}^0[G; I, \Lambda; P]$ is a rectangular 0-band $T \cong S/\mathcal{H}$ (associated homomorphism is $\natural : (i, g, \lambda) \mapsto (i, \lambda)$).

Rectangular 0-band example



$$P = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$$A = \{(1,1),(2,3),(3,4),(4,2)\}$$

$$\begin{aligned} (1,1)(2,3) &= (1,3) \text{ since } p_{12} = 1 \\ (2,3)(1,1) &= 0 \qquad \text{ since } p_{31} = 0 \end{aligned}$$

Rectangular 0-bands

Theorem (RG, Ruškuc (2005))

Let S be an $m \times n$ rectangular 0-band. Then

$$rank(S) = \max(m, n).$$

Proof. By induction on the dimensions of S, using regularity. □

Rectangular 0-bands

Theorem (RG, Ruškuc (2005))

Let S be an $m \times n$ rectangular 0-band. Then

$$\mathit{rank}(S) = \max(m, n).$$

Proof. By induction on the dimensions of S, using regularity. □

Corollary

Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a finite completely 0-simple semigroup. If S is idempotent generated then $\mathit{rank}(S) = \max(|I|, |\Lambda|)$.

Proof. Let $T = S\natural$. Let $\langle A \rangle = T$ with $|A| = \max(|I|, |\Lambda|)$. Let B be a transversal of the $\mathcal{H}$-classes $A\natural^{-1}$. Then $\langle B \rangle \cap H \neq \varnothing$ for every $\mathcal{H}$-class H of S. Therefore $\langle B \rangle \supseteq \langle E(S) \rangle = S$. □

Application

Recall that the proper ideals of T_n are given by $K(n,r) = \{\alpha \in T_n : |\mathrm{im}(\alpha)| \leq r\}$ where $1 \leq r < n$.

Lemma. $K(n,r)$ is idempotent generated.

Lemma. $K(n,r)$ is generated by the elements in its unique maximal $\mathcal{J}$-class J_r.

Corollary (Howie and McFadden (1990))

Let $n \in \mathbb{N}$ and let $1 < r < n$. Then:

$$rank(K(n,r)) = S(n,r).$$

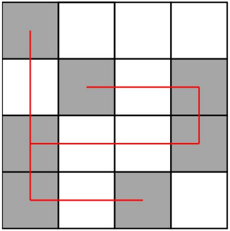
Proof: $\mathrm{rank}(K(n,r)) = \mathrm{rank}(J_r{}^*) = \max(S(n,r), \binom{n}{r}) = S(n,r)$. □

Associated graphs: $\Gamma(S)$

Definition Given $S = \mathcal{M}^0[G; I, \Lambda; P]$ we let $\Gamma(S)$ denote the graph with set of vertices $\{(i, \lambda) \in I \times \Lambda : H_{i\lambda} \text{ is a group}\}$ and (i, λ) adjacent to (j, μ) if and only if $i = j$ or $\lambda = \mu$.

We say $S = \mathcal{M}^0[G; I, \Lambda; P]$ is connected if $\Gamma(S)$ is connected.

Example Connected.

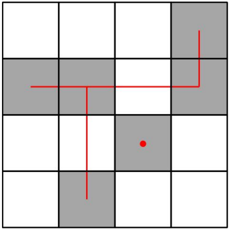


Associated graphs: $\Gamma(S)$

Definition Given $S = \mathcal{M}^0[G; I, \Lambda; P]$ we let $\Gamma(S)$ denote the graph with set of vertices $\{(i, \lambda) \in I \times \Lambda : H_{i\lambda} \text{ is a group}\}$ and (i, λ) adjacent to (j, μ) if and only if $i = j$ or $\lambda = \mu$.

We say $S = \mathcal{M}^0[G; I, \Lambda; P]$ is connected if $\Gamma(S)$ is connected.

Example Disconnected.



Products of idempotents

The set of idempotents of S is

$$E(S) = \{(i, {p_{\lambda i}}^{-1}, \lambda) : i \in I,\ \lambda \in \Lambda,\ p_{\lambda i} \neq 0\}.$$

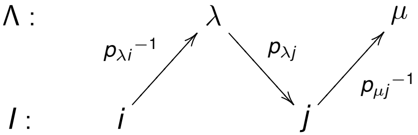
Provided $p_{\lambda j} \neq 0$ we have:

$$(i, {p_{\lambda i}}^{-1}, \lambda)(j, {p_{\mu j}}^{-1}, \mu) = (i, {p_{\lambda i}}^{-1} p_{\lambda j} {p_{\mu j}}^{-1}, \mu)$$

which can be written as:

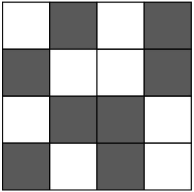
$$i \xrightarrow{{p_{\lambda i}}^{-1}} \lambda \xrightarrow{p_{\lambda j}} j \xrightarrow{{p_{\mu j}}^{-1}} \mu\ .$$

Grouping together elements of I and those of Λ gives:

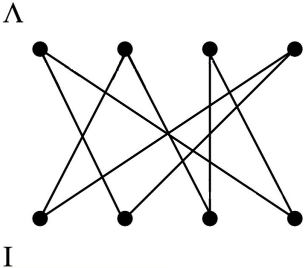


The graph $\Delta(P)$

Definition. Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a completely 0-simple semigroup. We let $\Delta(P)$ denote the undirected bipartite graph with set of vertices $I \cup \Lambda$ and an edge between i and λ if and only if $p_{\lambda i} \neq 0$.



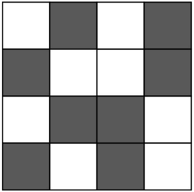
$S = \mathcal{M}^0[G; I, \Lambda; P]$



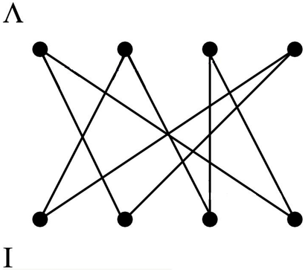
$\Delta(P)$

The graph $\Delta(P)$

Definition. Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a completely 0-simple semigroup. We let $\Delta(P)$ denote the undirected bipartite graph with set of vertices $I \cup \Lambda$ and an edge between i and λ if and only if $p_{\lambda i} \neq 0$.



$S = \mathcal{M}^0[G; I, \Lambda; P]$



$\Delta(P)$

Lemma. The following conditions are equivalent:

(i) $\Gamma(S)$ is a connected graph;

(ii) $\Delta(P)$ is a connected graph;

(iii) $\langle E(S) \rangle \cap H_{i\lambda} \neq \varnothing$ for any $i \in I$ and any $\lambda \in \Lambda$.

Paths and values

There is a natural correspondence between non-zero products of idempotents in S and paths in $\Delta(P)$ starting in I and ending in Λ.

Definition

In $\Delta(P)$ the value of the path $\pi = z_1 \to z_2 \to \ldots \to z_t$ is defined to be

$$V(\pi) = \phi(z_1, z_2)\phi(z_2, z_3)\ldots\phi(z_{t-1}, z_t) \in G$$

where

$$\phi(i, \lambda) = p_{\lambda i}^{-1}, \quad \phi(\lambda, i) = p_{\lambda i}, \quad i \in I, \quad \lambda \in \Lambda.$$

Let $x, y \in I \cup \Lambda$.

- $\mathcal{P}_{x,y}$ - all paths in $\Delta(P)$ with initial vertex x and terminal vertex y.
- $V_{x,y} = \{V(\pi) : \pi \in \mathcal{P}_{x,y}\}$ - values of paths from x to y.

Relative Rank

Definition

Let S be a semigroup and let T be a subset of S.

The *relative rank* of S modulo T is the minimum number of elements of S that are need to be added to T in order to generate the whole of S:

$$\mathrm{rank}(S : T) = \min\{|A| : A \subseteq S, \langle T \cup A \rangle = S\}.$$

Example

1. $\mathrm{rank}(S : S) = 0$, $\mathrm{rank}(S : \varnothing) = \mathrm{rank}(S)$.
2. $\mathrm{rank}(T_n : S_n) = 1$, $\mathrm{rank}(P_n : T_n) = 1$.

Main result (I)

Theorem. (RG & Ruškuc (2005)) Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a finite Rees matrix semigroup with k connected components $I_1 \times \Lambda_1, \ldots, I_k \times \Lambda_k$.

- For every $j = 1, \ldots, k$ choose $(1_{I_j}, 1_{\Lambda_j}) \in I_j \times \Lambda_j$ with $p_{1_{\Lambda_j} 1_{I_j}} \neq 0$.

Main result (I)

Theorem. (RG & Ruškuc (2005)) Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a finite Rees matrix semigroup with k connected components $I_1 \times \Lambda_1, \ldots, I_k \times \Lambda_k$.

- For every $j = 1, \ldots, k$ choose $(1_{I_j}, 1_{\Lambda_j}) \in I_j \times \Lambda_j$ with $p_{1_{\Lambda_j} 1_{I_j}} \neq 0$.
- For $\lambda \in \Lambda_r$ and $i \in I_l$ define:
  - (i) π_λ - a path connecting 1_{I_r} to λ in the subgraph $I_r \cup \Lambda_r$;
  - (ii) π_i - a path connecting i to 1_{Λ_l} in the subgraph $I_l \cup \Lambda_l$.

Main result (I)

Theorem. (RG & Ruškuc (2005)) Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a finite Rees matrix semigroup with k connected components $I_1 \times \Lambda_1, \ldots, I_k \times \Lambda_k$.

- For every $j = 1, \ldots, k$ choose $(1_{I_j}, 1_{\Lambda_j}) \in I_j \times \Lambda_j$ with $p_{1_{\Lambda_j} 1_{I_j}} \neq 0$.
- For $\lambda \in \Lambda_r$ and $i \in I_l$ define:
  (i) π_λ - a path connecting 1_{I_r} to λ in the subgraph $I_r \cup \Lambda_r$;
  (ii) π_i - a path connecting i to 1_{Λ_l} in the subgraph $I_l \cup \Lambda_l$.
- For every $r = 1, \ldots, k$ let

$$a_{\lambda i} = V(\pi_\lambda) p_{\lambda i} V(\pi_i) p_{1_{\Lambda_r} 1_{I_r}} \quad ((\lambda, i) \in \Lambda_r \times I_r).$$

$H_r :=$ subgroup generated by the set $\{a_{\lambda i} \mid (\lambda, i) \in \Lambda_r \times I_r, a_{\lambda i} \neq 0\}$.

Main result (I)

Theorem. (RG & Ruškuc (2005)) Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a finite Rees matrix semigroup with k connected components $I_1 \times \Lambda_1, \ldots, I_k \times \Lambda_k$.

- For every $j = 1, \ldots, k$ choose $(1_{I_j}, 1_{\Lambda_j}) \in I_j \times \Lambda_j$ with $p_{1_{\Lambda_j} 1_{I_j}} \neq 0$.
- For $\lambda \in \Lambda_r$ and $i \in I_l$ define:
  - (i) π_λ - a path connecting 1_{I_r} to λ in the subgraph $I_r \cup \Lambda_r$;
  - (ii) π_i - a path connecting i to 1_{Λ_l} in the subgraph $I_l \cup \Lambda_l$.
- For every $r = 1, \ldots, k$ let

$$a_{\lambda i} = V(\pi_\lambda) p_{\lambda i} V(\pi_i) p_{1_{\Lambda_r} 1_{I_r}} \ \ ((\lambda, i) \in \Lambda_r \times I_r).$$

$H_r :=$ subgroup generated by the set $\{a_{\lambda i} \mid (\lambda, i) \in \Lambda_r \times I_r, a_{\lambda i} \neq 0\}$.

Then

$$\text{rank}(S) = \max(|I|, |\Lambda|, \rho_{\min} + k - 1),$$

where

$$\rho_{\min} = \min\{\text{rank}(G : \bigcup_{i=1}^{k} g_i H_i {g_i}^{-1}) | g_1, \ldots, g_k \in G\}.$$

Graham normal form (R. L. Graham 1968)

Let $S = \mathcal{M}^0[G; I, \Lambda; P]$. It is always possible to normalize the structure matrix P to obtain Q with the following properties:

1. the matrix Q is a direct sum of r blocks $C_1, \ldots, C_r$:

$$\begin{array}{c} \\ \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_r \end{array} \begin{array}{c} \begin{array}{cccc} I_1 & I_2 & \ldots & I_r \end{array} \\ \begin{pmatrix} C_1 & & & 0 \\ & C_2 & & \\ & & \ddots & \\ 0 & & & C_r \end{pmatrix} \end{array}.$$

2. Each matrix $C_i : \Lambda_i \times I_i \to G^0$ is regular and

$$\langle E(S) \rangle = \bigcup_{i=1}^{r} \mathcal{M}^0[G_i; I_i, \Lambda_i; C_i]$$

where G_i is the subgroup of G generated by the non-zero entries of C_i, for $i = 1, \ldots, r$.

Main result (II)

Theorem (RG & Ruškuc (2005))

- $S = \mathcal{M}^0[G; I, \Lambda; P]$ - *a finite Rees matrix semigroup with k connected components* $I_1 \times \Lambda_1, \ldots, I_k \times \Lambda_k$.
- P - *regular matrix in Graham normal form.*
- *For* $r = 1, \ldots, k$ *let* H_r *be the subgroup of* G *generated by the non-zero entries of component* $C_r = I_r \times \Lambda_r$ *of* P.

Then

$$rank(S) = max(|I|, |\Lambda|, \rho_{min} + k - 1)$$

where

$$\rho_{min} = \min\{rank(G : \bigcup_{i=1}^{k} g_i H_i {g_i}^{-1}) \mid g_1, \ldots, g_k \in G\}.$$

Corollaries

A *Hamiltonian* group is a group all of whose subgroups are normal. In particular, all abelian groups are Hamiltonian.

Corollary

Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ *be a finite Rees matrix semigroup with* G *a Hamiltonian group,* k *connected components and with regular matrix* P *in Graham normal form. Let* H *be the subgroup of* G *generated by the non-zero entries of* P*. Then*

$$rank(S) = max(|I|, |\Lambda|, rank(G : H) + k - 1).$$

Proof. Conjugating has no effect on subgroups □

Brandt semigroups

Definition

- A semigroup S is *regular* if every $\mathcal{R}$-class and every $\mathcal{L}$-class of S contains at least one idempotent.
- A semigroup S is *inverse* if every $\mathcal{R}$- and every $\mathcal{L}$-class contains exactly one idempotent.
- A Brandt semigroup $B = B(G, n)$ is a Rees matrix semigroup $\mathcal{M}^0[G; I, I; P]$ where $P \sim I_n$, the $n \times n$ identity matrix, and $I = \{1, \ldots, n\}$.

Fact. A finite 0-simple semigroup is inverse iff it is isomorphic to $B(G, n)$ for some group G and some $n \in \mathbb{N}$.

Brandt semigroups

Corollary

Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a finite Rees matrix semigroup with k connected components and with regular matrix P only containing entries from $\{0, 1\}$. Then

$$rank(S) = max(|I|, |\Lambda|, rank(G) + k - 1).$$

Proof

$$\rho_{\min} = \min\{\text{rank}(G : \bigcup_{i=1}^{k} g_i H_i {g_i}^{-1}) | g_1, \ldots, g_k \in G\} = \text{rank}(G : \{1_G\}) = \text{rank}(G).$$

□

Corollary (Gomes and Howie (1987))

Let $B = B(G, n)$ be a Brandt semigroup, where G is a finite group of rank $r \geq 1$. Then the rank of B is $r + n - 1$.

Ideals of symmetric inverse semigroup

The ideals of symmetric inverse semigroup I_n are the sets $L(n,r) = \{\alpha \in I_n : |\mathrm{im}(\alpha)| \leq r\}$.

Corollary (Garba (1994))

Let $n \in \mathbb{N}$ and let $1 < r < n$. Then:

$$rank(L(n,r)) = \binom{n}{r} + 1.$$

Proof

$$\mathrm{rank}(L(n,r)) = \mathrm{rank}(L(n,r)/L(n,r-1)) = \mathrm{rank}(B(S_r, \binom{n}{r})) = 2 + \binom{n}{r} - 1 \quad \square$$

Note. The ranks of the two-sided ideals of all the other semigroups of transformations mentioned earlier can also be recovered as applications of the main theorem.

Subsemigroups of T_n

Definition

- Let $n, r \in \mathbb{N}$ with $2 < r < n$.
- Let A be a set of r-subsets of $\{1, \ldots, n\}$.
- Let B be a set of partitions of $\{1, \ldots, n\}$, each with r classes.

Let:

$$S(A, B) = \langle \, \{\alpha \in T_n : \mathrm{im}(\alpha) \in A, \; \ker \alpha \in B\} \, \rangle,$$

and let $\Gamma(A, B) = A \cup B$ be the bipartite graph where $\mathcal{I} \in A$ is connected to $\mathcal{K} \in B$ iff $\mathcal{I}$ is a transversal of $\mathcal{K}$.

Example

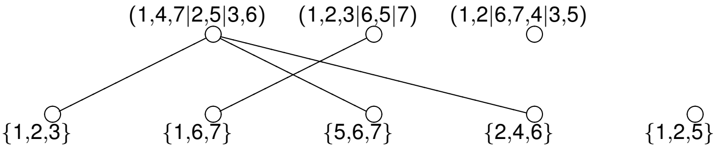
Let $n = 7$ and $r = 3$ and define the set of images:

$$A = \{\{1, 2, 3\}, \{1, 6, 7\}, \{5, 6, 7\}, \{2, 4, 6\}, \{1, 2, 5\}\}$$

and set of partitions:

$$B = \{(1, 4, 7|2, 5|3, 6), (1, 2, 3|6, 5|7), (1, 2|6, 7, 4|3, 5)\}.$$

Then the graph $\Gamma(A, B)$ is isomorphic to:



Family of transformation semigroups

Theorem (RG 2005)

- *Let* $n, r \in \mathbb{N}$ *with* $2 < r < n$.
- A *- a set of* r*-subsets of* $\{1, \ldots, n\}$.
- B *- a set partitions of* $\{1, \ldots, n\}$ *each with* r *classes.*

Then

$$rank(S(A,B)) = \begin{cases} \max(v_+(A), v_+(B)) + v_0 & \text{if } \mathcal{MD} \geq 2 \\ \max(v_+(A), v_+(B)) + v_0 + 1 & \text{if } \mathcal{MD} = 1 \\ |A||B|r! & \text{if } \mathcal{MD} = 0 \end{cases}$$

where

- $v_+(X) := |\{x \in X : d(x) > 0\}|; \quad v_0 := |\{v \in \Gamma(A,B) : d(v) = 0\}|$
- $\mathcal{MD} = \max\{d(v) : v \in \Gamma(A,B)\}$.

Example

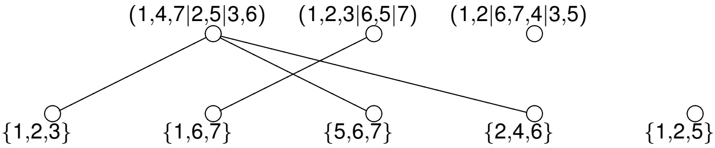
Let $n = 7$ and $r = 3$ and define the set of images:

$$A = \{\{1,2,3\},\{1,6,7\},\{5,6,7\},\{2,4,6\},\{1,2,5\}\}$$

and set of partitions:

$$B = \{(1,4,7|2,5|3,6),(1,2,3|6,5|7),(1,2|6,7,4|3,5)\}.$$

Then the graph $\Gamma(A, B)$ is isomorphic to:



which has two isolated vertices so that $v_0 = 2$, $v_+(B) = 2$, $v_+(A) = 4$ and maximum degree $\mathcal{MD} = 3$. Therefore:

$$\text{rank}(S(A,B)) = \max(2,4) + 2 = 6.$$

Final application

Theorem (McIver & Neumann (1987))

rank$(G) \leq \max(2, \lfloor n/2 \rfloor)$ *for all* $G \leq S_n$ *(and this is best possible).*

Theorem (RG 2005)

Let $n \geq 4$ *and let* $1 < r < n$. *Every regular subsemigroup of* T_n *that is generated by mappings all with image size equal to* r, *and has a unique maximal* $\mathcal{J}$*-class, is generated by at most* $S(n, r)$ *elements. Moreover, the bound is attained by the semigroup* $K(n, r)$.

Open problems

- Can the above results for completely 0-simple semigroups be extended to:
  - finitely generated completely 0-simple semigroups?
  - Rees matrix semigroups over arbitrary monoids?
  - semigroups with more complicated ideal structure (e.g. small monoids)?

- Prove the analogue of McIver and Neumann's theorem for subsemigroups of the full transformation semigroup (i.e. determine $\max\{\text{rank}(S) : S \leq T_n\}$).