# Finiteness conditions and index in semigroup theory

Robert Gray

University of Leeds

Glasgow, November 2006

# Outline

# Semigroups and finiteness conditions

## Definition

A *semigroup* is a pair $S = (S, \cdot)$ where $S$ is a set and $\cdot$ is a binary operation satisfying the associative law.

**Examples**

- Groups
- Subsemigroups of groups (e.g. $(\mathbb{N}, +) \leq (\mathbb{Z}, +)$)
- Semigroups of transformations
- Free semigroup $A^+$ over (finite) alphabet $A$

## Definition

A property $\mathcal{P}$ of semigroups is a *finiteness condition* if all finite semigroups satisfy $\mathcal{P}$. (e.g. being finite, finitely generated, finitely presented, locally finite, etc. )

# Inheritance of properties

Let $S$ be a semigroup with $T$ a subsemigroup of $S$.

Let $\mathcal{P}$ be a property of semigroups.

- $S$ satisfies $\mathcal{P} \Rightarrow T$ satisfies $\mathcal{P}$?
- $T$ satisfies $\mathcal{P} \Rightarrow S$ satisfies $\mathcal{P}$?

# What is index?

Let $S$ be a semigroup and let $T$ be a subsemigroup of $S$.

Roughly speaking...

Index is a measure of the ‘size’ of $T$ inside $S$.

A ‘good’ definition of index should have the property that if $T$ is ‘big’ in $S$ then $S$ and $T$ share many properties.

# Group index

$G$ - a group,  $H$ - subgroup of $G$

- $[G : H]$ is the number of (right) cosets of $H$ in $G$.
- A subgroup of finite index "‘differs by a finite amount from the group"’.

## Facts

Let $G$ be a group, let $H$ be a subgroup of finite index in $G$, and let $\mathcal{P}$ be any of the following conditions:

- finitely generated
- periodic
- residually finite
- $FP_n$, $FDT$
- finitely presented
- locally finite
- soluble word problem
- automatic.

Then $G$ satisfies $\mathcal{P}$ if and only if $H$ satisfies $\mathcal{P}$.

# Rees index

$S$ - a semigroup, $T$ - a subsemigroup of $S$

## Definition (Rees index)

The Rees index $[S:T]_R$ is the size of the complement $S \setminus T$.

## Facts

Let $S$ be a semigroup, let $T$ be a subsemigroup of finite Rees index in $S$, and let $\mathcal{P}$ be any of the following conditions:

- finitely generated / presented
- periodic
- residually finite
- locally finite
- soluble word problem
- automatic.

Then $S$ satisfies $\mathcal{P}$ if and only if $T$ satisfies $\mathcal{P}$.

(Jura (1978), Ruškuc (1998), Ruškuc & Thomas (1998), Hoffmann, Thomas, Ruškuc (2002), RG & Ruškuc (2006))

# Rees vs. group index

- Rees index behaves very much like group index.
- Rees index does not generalise group index.
- Rees index is a very restrictive condition.

## General problem 1

Find a notion of index that is weaker than Rees index but still maintains the “nice” properties of Rees index.

## General problem 2

Find a definition of index that:

- generalises both Rees and group index;
- maintains the "‘nice"’ properties of Rees and group index.

# Answering problem 1

**Weakening Rees index**

**Boundaries in Cayley graphs**

# Cayley Graphs

## Definition

Let $S$ be a semigroup generated by a finite set $A$.

The right Cayley graph $\Gamma_r(A, S)$ has:

- Vertices: elements of $S$.
- Edges: directed and labelled with letters from $A$.

$$s \xrightarrow{a} t \Leftrightarrow sa = t$$

- Given an edge $e = s \xrightarrow{a} t$ we define

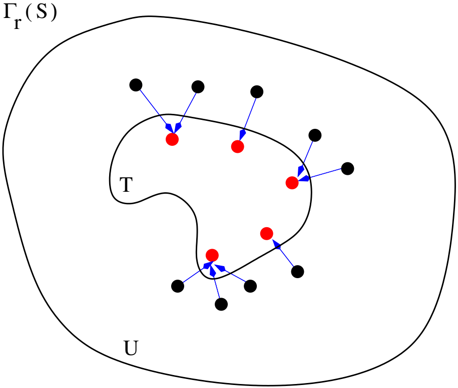
$$\iota(e) = s, \quad \tau(e) = t$$

calling them the initial and terminal vertices of $e$.

# Weakening Rees index

The general idea:

Restrict the number of points where $T$ and $S \setminus T$ "meet each other" in the Cayley graph to be finite.




- Red vertices

The "boundary" of $T$ in $S$.

# Semigroup boundaries

## Definition

- Let $T$ be a subsemigroup of $S$, where $S = \langle A \rangle$.
- The right boundary of $T$ in $S$ is the set of elements of $T$ that *receive an edge from* $S \setminus T$ in the right Cayley graph of $S$:

$$\mathcal{B}_r(A, T) = (S \setminus T)A \cap T.$$

# Semigroup boundaries

## Definition

- Let $T$ be a subsemigroup of $S$, where $S = \langle A \rangle$.
- The right boundary of $T$ in $S$ is the set of elements of $T$ that *receive an edge from* $S \setminus T$ in the right Cayley graph of $S$:

$$\mathcal{B}_r(A, T) = (S \setminus T)A \cap T.$$

- The left boundary of $T$ in $S$ is the set of elements of $T$ that receive an edge from $S \setminus T$ in the left Cayley graph of $S$:

$$\mathcal{B}_l(A, T) = A(S \setminus T) \cap T.$$

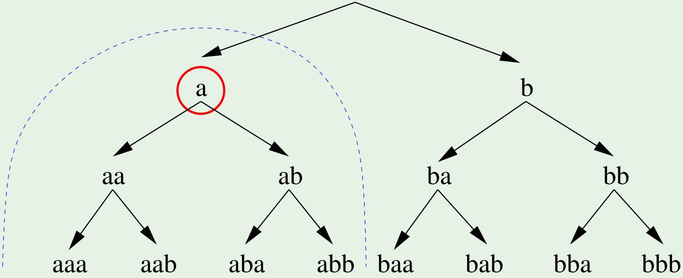
- The (two-sided) boundary is the union of the left and right boundaries:

$$\mathcal{B}(A, T) = \mathcal{B}_l(A, T) \cup \mathcal{B}_r(A, T).$$

# A straightforward example

## Example (Free monoid on two generators)

- $S = \{a, b\}^*$, $T = \{\text{words that begin with the letter } a\}$.

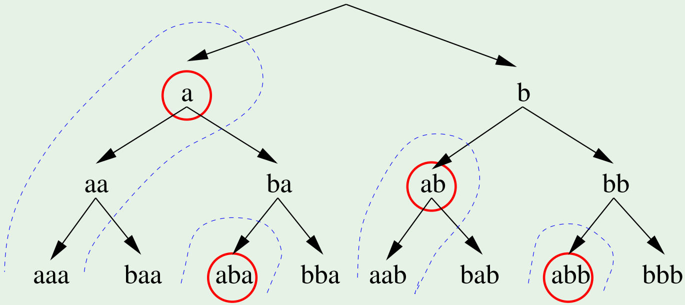



Right boundary: $\mathcal{B}_r(\{a, b\}, T) = \{a\}$.

# A straightforward example

## Example (Free monoid on two generators)

- $S = \{a, b\}^*$, $\quad T = \{$words that begin with the letter $a\}$.




Left boundary: $\mathcal{B}_l(\{a, b\}, T) = \{a\} \cup \{ab\{a, b\}^*\}$.

# Generators and relations

## Facts

- The finiteness of $\mathcal{B}(A, T)$ is independent of the choice of finite generating set $A$ of $S$.
- If $T$ has finite Rees index in $S$ then $T$ has a finite boundary in $S$.

## Theorem (RG, Ruškuc (2006))

*If $S$ is a finitely generated semigroup and $T$ is a subsemigroup of $S$ with finite boundary then $T$ is finitely generated.*

# Generators and relations

## Facts

- The finiteness of $\mathcal{B}(A, T)$ is independent of the choice of finite generating set $A$ of $S$.
- If $T$ has finite Rees index in $S$ then $T$ has a finite boundary in $S$.

## Theorem (RG, Ruškuc (2006))

*If $S$ is a finitely generated semigroup and $T$ is a subsemigroup of $S$ with finite boundary then $T$ is finitely generated.*

## Theorem (RG, Ruškuc (2006))

*Let $S$ be a finitely generated semigroup and $T$ be a subsemigroup of $S$. If $S$ is a finitely presented and $T$ has a finite boundary in $S$ then $T$ is finitely presented.*

# Boundaries

**Good things**

- Much weaker than Rees index (so answers Problem 1)
- Provides common framework for finite Rees index and ideal complement results
- Corrects a mistake in the original finite Rees index proof

**Bad things**

- Does not provide a common generalisation of group and Rees index (so does not answer Problem 2);
- Is one-directional, only allows results of the form:

$$S \text{ has } \mathcal{P} \Rightarrow T \text{ has } \mathcal{P}.$$

- Not interesting for properties inherited by all substructures (e.g. locally finite, residually finite, soluble word problem etc.)

# Higher dimensions - Finite derivation type

- FDT is a property of finitely presented semigroups.
- Can be thought of as a higher dimensional version of the property of being finitely presented, think “relations between relations”
- Originated from the study of finite complete string-rewriting systems

## Open problem

Let $S$ be a finitely presented semigroup, $T \leq S$ with finite boundary.

If $S$ has FDT then does $T$ have FDT?

- If $T$ is an ideal with finite Rees index then $S$ has FDT implies $T$ has FDT (Malheiro 2006)
- If $S$ is free then $T$ has FDT (RG, Pride (work in progress))

# Answering problem 2

**Common generalisation of Rees and group index**

**Attempt 1: Syntactic index**

# Syntactic index (Ruškuc & Thomas (1998))

## Definition

Let $S$ be a semigroup and let $T$ be a subsemigroup of $S$.

The (right) syntactic congruence corresponding to $T$ is the largest right congruence $\rho$ on $S$ such that $T$ is a union of congruence classes:

$$\Sigma_r(T) = \{(x, y) \in S \times S : (\forall s \in S^1)(xs \in T \Leftrightarrow ys \in T)\}.$$

The number of $\Sigma_r(T)$-classes is called the (right) syntactic index of $T$ in $S$, and it denoted $[S : T]_S$.

# Syntactic index - examples

## Example

- If $G$ is a group and $H$ is a subgroup then

  $$(x, y) \in \Sigma_r(H) \Leftrightarrow x \& y \text{ belong to the same right coset of } H \text{ in } G.$$

  The largest right congruence on $G$ such that $H$ is a union of congruence classes is the one that has the right cosets of $H$ as its congruence classes.

- Let $S$ be a semigroup and let $T$ be a subsemigroup of $S$. If $T$ has finite Rees index in $S$ then $T$ has finite syntactic index in $S$.

## Conclusion

Syntactic index provides a common generalisation of group and Rees index.

# Syntactic index is too weak

**Theorem (Ruškuc and Thomas (1998))**

- *$\mathcal{P}$ - a non-trivial property of semigroups*
- *There is a semigroup $S$ with a subsemigroup $T$ of finite syntactic index such that either*
  1. *$S$ satisfies $\mathcal{P}$ and $T$ does not satisfy $\mathcal{P}$; or*
  2. *$S$ does not satisfy $\mathcal{P}$ and $T$ satisfies $\mathcal{P}$.*

**Corollary.** Finiteness, periodicity, local finiteness, and residual finiteness are not inherited by syntactically small extensions.

**Fact**

Neither of the properties of being finitely generated or being finitely presented are inherited by either syntactically small extensions or syntactically large subsemigroups.

# Answering problem 2

**Common generalisation of Rees and group index**

**Attempt 2: Green index**

# Green's relations

According to Wikipedia Green's relations:

- are 5 equivalence relations that characterize the elements of a semigroup in terms of the principal ideals they generate.
- They are...

  *"so all-pervading that, on encountering a new semigroup, almost the first question one asks is 'What are the Green relations like?' " (J. M. Howie 2002)*

# Green's relations

$S$ - semigroup, $a, b \in S$.

- $S^1 a = Sa \cup \{a\}$ is the smallest left ideal of $S$ containing $a$. It is called the principal left ideal generated by $a$.
- $aS^1 = aS \cup \{a\}$ - the principal right ideal generated by $a$.

## Definition (Green's $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{H}$ relations)

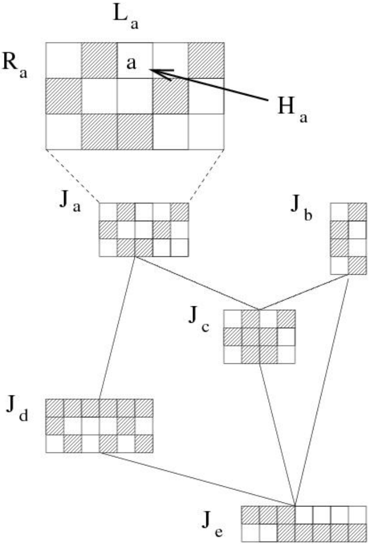
$$a\mathcal{L}b \Leftrightarrow S^1 a = S^1 b, \qquad a\mathcal{R}b \Leftrightarrow aS^1 = bS^1, \qquad \mathcal{H} = \mathcal{R} \cap \mathcal{L}.$$

## Facts

- $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{H}$ are equivalence relations;
- $\mathcal{L}$ is a right congruence, $\mathcal{R}$ is a left congruence;
- The $\mathcal{R}$-classes are the *strongly connected components* of the right Cayley graph (dual for left);
- If $S$ is a group then $\mathcal{L} = \mathcal{R} = \mathcal{H}$.

# Green's relations

## Important tool for structure theory




$S$ - semigroup, $x, y \in S$

$$x\mathcal{R}y \Leftrightarrow xS^1 = yS^1$$
$$x\mathcal{L}y \Leftrightarrow S^1x = S^1y$$
$$x\mathcal{J}y \Leftrightarrow S^1xS^1 = S^1yS^1$$

- $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$
- $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$
- $J_x \leq J_y \Leftrightarrow S^1xS^1 \subseteq S^1yS^1$

# Relative Green's relations

$S$ - semigroup, $T$ - subsemigroup of $S$, $a, b \in S$.

- $T^1 a = Ta \cup \{a\}$ : $T$-relative principal left ideal generated by $a$ in $S$.
- $aT^1 = aT \cup \{a\}$ : $T$-relative principal right ideal generated by $a$ in $S$.

## Definition (Relative Green's relations)

$$a\mathcal{L}^T b \Leftrightarrow T^1 a = T^1 b, \qquad a\mathcal{R}^T b \Leftrightarrow aT^1 = bT^1, \qquad \mathcal{H}^T = \mathcal{R}^T \cap \mathcal{L}^T.$$

## Facts

- $\mathcal{L}^T$, $\mathcal{R}^T$, and $\mathcal{H}^T$ are equivalence relations;
- The $\mathcal{R}^T$-classes are the *strong orbits* of $T^1$ acting on $S$ by right multiplication (dual for left);
- $T$ is a union of $\mathcal{H}^T$-classes.

Wallace (1962) - developed a theory of relative Green's relations.

# Green index

## Definition and examples

**Definition**

Let $\{H_j : j \in J\}$ be the $\mathcal{H}^T$-classes of $S \setminus T$. Then we define:

$$[S : T]_G = |J| + 1$$

and call it the Green index of $T$ in $S$.

**Examples.**

- $G$ - group, $H$ - subgroup of $G$.
  - $u\mathcal{R}^H v \Leftrightarrow uH = vH$ (left cosets of $H$ in $G$)
  - $u\mathcal{L}^H v \Leftrightarrow Hu = Hv$ (right cosets of $H$ in $G$)
  - Therefore if $[G : H] < \infty$ then $[G : H]_G < \infty$.
- $S$ - semigroup, $T$ - subsemigroup of $S$.
  - If $T$ has finite Rees index in $S$ then the number of $\mathcal{H}^T$-classes of $S \setminus T$ is at most $|S \setminus T|$.

# Examples and basic properties

- If $U \leq T \leq S$ then $[S : U]_G$ is finite if and only if both $[T : U]_G$ and $[S : T]_G$ are finite.

- If $[S : T]_G$ is finite then $S$ is finite if and only if $T$ is finite.

- If $T \leq S$ is an ideal then $T$ has finite Green index in $S$ if and only if $T$ has finite Rees index in $S$.

- If $S$ is an inverse semigroup (in particular if $S$ is a group) then the following are equivalent:
  1. $T$ has finite Green index in $S$;
  2. $S \setminus T$ has finitely many $\mathcal{R}^T$-classes;
  3. $S \setminus T$ has finitely many $\mathcal{L}^T$-classes.

# The lemmas of Schreier and Jura

## Lemma (Schreier)

- *$G$ - a group generated by $A$, $H$ - finite index subgroup of $G$*
- *$K$ be coset representatives (with $1 \in K$)*
- *$\overline{g}$ denotes the coset representative of $g \in G$*

*Then $H$ is generated by:*

$$X = \{ka(\overline{ka})^{-1} : k \in K, a \in A\}.$$

## Lemma (Jura (1978))

- *$S$ - a semigroup generated by $A$*
- *$T$ - finite Rees index subsemigroup of $S$*

*Then $T$ is generated by:*

$$X = \{s_1 a s_2 : s_1, s_2 \in S^1 \setminus T,\ a \in A,\ s_1 a, s_1 a s_2 \in T\}.$$

# A generating set for $T$

$S$ - semigroup, $T$ - subsemigroup with finite Green index.

- $h_i : i \in I$ - representatives of the $\mathcal{H}^T$-classes of $S \setminus T$;
- Define $R_1 = L_1 = H_1 = \{1\}$, $\quad r_1 = l_1 = h_1 = 1$.
- For $s \in S$ and $i \in I$ define:

$$\rho(s,i) = \begin{cases} j & \text{if } sh_i \in H_j \\ 1 & \text{if } sh_i \in T \end{cases}, \qquad \lambda(i,s) = \begin{cases} j & \text{if } h_i s \in H_j \\ 1 & \text{if } h_i s \in T. \end{cases}$$

- Moreover, let $\sigma(s,i) \in T$ and $\tau(i,s) \in T$ satisfy:

$$sh_i = h_{\rho(s,i)}\sigma(s,i), \qquad h_i s = \tau(i,s)h_{\lambda(i,s)}.$$

**Lemma**

*If $S = \langle A \rangle$ then $T$ is generated by the set:*

$$X = \{\tau(i, \sigma(a,j)) \ : \ i,j \in I \cup \{1\}, \ a \in A\}.$$

# Finite generation

**Corollary (RG and Ruškuc (in preparation))**

*Let $S$ be a semigroup and let $T$ be a subsemigroup of $S$ with finite Green index. Then $S$ is finitely generated if and only if $T$ is finitely generated.*

**Fact.** There is an example of a semigroup $S$ with subsemigroup $T$:

- $S$ is finitely generated;
- $S \setminus T$ has finitely many relative $\mathcal{R}$-classes;
- $T$ is not finitely generated.

# Other properties

**Theorem (RG and Ruškuc (in preparation))**

*Let $S$ be a semigroup, let $T$ be a subsemigroup of finite Green index in $S$.*

*Let $\mathcal{P}$ be any of the following conditions:*

- *periodic*
- *locally finite*
- *finitely many left (resp. right) ideals*
- *finitely many idempotents*

*Then $S$ satisfies $\mathcal{P}$ if and only if $T$ satisfies $\mathcal{P}$.*

# Other properties

**Theorem (RG and Ruškuc (in preparation))**

*Let $S$ be a semigroup, let $T$ be a subsemigroup of finite Green index in $S$.*

*Let $\mathcal{P}$ be any of the following conditions:*

- *periodic*
- *locally finite*
- *finitely many left (resp. right) ideals*
- *finitely many idempotents*

*Then $S$ satisfies $\mathcal{P}$ if and only if $T$ satisfies $\mathcal{P}$.*

**Question**

Can we prove the same result for the properties of being finitely presented, residually finite, or having soluble word problem?

# Clifford monoid example

## Example

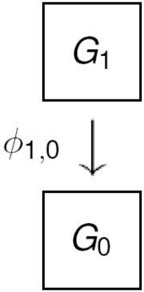
- $Y = \{0, 1\}$ with $0 < 1$, a 2 element semilattice

- $G_1 := \langle A \mid \ \rangle$ free group where $|A| = r$
- $G_0 := \langle A \mid R \rangle$ a non-finitely presented homomorphic image of $G_1$

- $\phi_{1,0} : G_1 \to G_0$ : associated homomorphism
- $\phi_{i,i} : G_i \to G_i$ : identity maps

- Define multiplication on $S = G_0 \cup G_1$ by:

$$xy = (x\phi_{\alpha,\alpha\beta})(y\phi_{\beta,\alpha\beta}), \quad x \in S_\alpha, y \in S_\beta.$$



# Clifford monoid example

## Proposition

*Let $S = G_0 \cup G_1$ (above) and let $T = G_1 \leq S$. Then:*

1. *$T$ is finitely presented;*
2. *$T$ has Green index 2 in $S$;*
3. *$S$ is not finitely presented.*

**Proof.** $G_0$ is a retract of $S$. Since $G_0$ is not finitely presented it follows that neither is $S$. □

# Clifford monoid example

## Proposition

*Let $S = G_0 \cup G_1$ (above) and let $T = G_1 \leq S$. Then:*

1. *$T$ is finitely presented;*
2. *$T$ has Green index 2 in $S$;*
3. *$S$ is not finitely presented.*

**Proof.** $G_0$ is a retract of $S$. Since $G_0$ is not finitely presented it follows that neither is $S$. □

Using the same kind of construction we can prove:

1. Residual finiteness is not inherited by finite Green index extensions.
2. Having a soluble word problem is not inherited by finite Green index extensions. (For finitely generated semigroups.)

# Generalised Schützenberger groups

## Definition

- $T \leq S$, $H$ - any $\mathcal{H}^T$-class of $S$.
- (Wallace (1962)) TFAE
  - $H^2 \cap H \neq \varnothing$;
  - $H$ contains an idempotent ($e^2 = e$);
  - $H$ is a subgroup of $S$.

# Generalised Schützenberger groups

## Definition

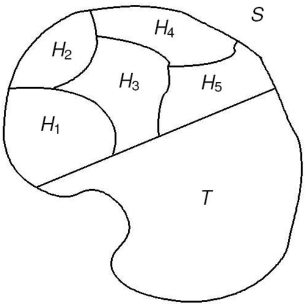
- $T \leq S$, $H$ - any $\mathcal{H}^T$-class of $S$.
- (Wallace (1962)) TFAE
  - $H^2 \cap H \neq \varnothing$;
  - $H$ contains an idempotent ($e^2 = e$);
  - $H$ is a subgroup of $S$.
- If $H$ is not a group then we can still associate a group with $H$:
  - $T(H) = \{t \in T : Ht = H\}$ : the stabilizer of $H$ in $T$.
  - The relation $\sim$ on $T(H) \leq T$ defined by:

$$x \sim y \Leftrightarrow (\forall h \in H)(hx = hy)$$

    is a congruence.
  - $\Gamma(H) = T(H)/\sim$ is a group, called the generalised Schützenberger group of $H$.
- $|\Gamma(H)| = |H|$ and if $H$ is a group then $\Gamma(H) \cong H$.

# A picture of what is going on

- $S$ - semigroup, $T$ subsemigroup of $S$
- $H_i : i \in I$ : the relative $\mathcal{H}$-classes in the complement
- $\Gamma_i$ a group associated with the set $H_i$ arising from the action of $T$ on $H_i$




**Question.** How are the properties of $S$ related to those of $T$ and the groups $\Gamma_i : i \in I$?

- If $T$ has finite Rees index in $S$ then all the groups $\Gamma_i$ are finite.
- If $S = G$ and $T = N \trianglelefteq G$ then $\Gamma_i \cong N$ for all $i \in I$.

In both cases $T$ has $\mathcal{P} \Rightarrow \Gamma_i$ has $\mathcal{P}$ for all $i \in I$.

# Finite presentability

**Theorem (RG and Ruškuc (in preparation))**

*Let $S$ be a semigroup and let $T$ be a subsemigroup of $S$ with finite Green index.*

*If $T$ is finitely presented and each group $\Gamma_i$ is finitely presented then $S$ is finitely presented.*

**Open problem.** Prove that if $S$ is finitely presented then $T$ is finitely presented and each group $\Gamma_i$ is finitely presented.

**Note.** We know that if $S$ is finitely generated then $T$ is finitely generated and each group $\Gamma_i$ is finitely generated.

# Syntactic index and residual finiteness

**Theorem (RG, Ruškuc (in preparation))**

*Let $S$ be a semigroup and let $T$ be a subsemigroup of $S$. If $T$ has finite Green index in $S$ then $T$ has finite syntactic index in $S$.*

**Conjecture**

Let $S$ be a semigroup and let $T$ a subsemigroup of $S$ with finite Green index.

Then $S$ is residually finite if and only if $T$ is residually finite and all the Schützenberger groups $\Gamma_i$ are residually finite.

**Note.** The relationship with syntactic index should be used in the proof, just as it was in the case of finite Rees index.

# Counting subsemigroups

**Theorem (Hall (1949))**

*Let $G$ be a group and let $n \geq 1$ be an integer. If $G$ is finitely generated then it has only finitely many subgroups of index $n$.*

**Theorem (RG, Ruškuc (in preparation))**

*A finitely generated semigroup has only finitely many subsemigroups of any given Green index $n$.*

**Proof.** Makes use of the relationship between Green index and syntactic index.

# Future work

- Green index gives a framework for providing unifying proofs of group index and Rees index results.
- It gives us a better understanding of why similar results exist for group index and Rees index.

- Open problems
  - Prove the "hard" direction of the finite presentability result
  - Prove the corresponding results for
    - residual finiteness
    - soluble word problem
    - FDT
    - automaticity
    - etc...
- How about inverse semigroup presentations?