Homogeneous structures

Robert Gray

Centro de Álgebra
da Universidade de Lisboa

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UNIVERSIDADE
DE LISBOA

Group theory

A group is...

a set G with a binary operation $G \times G \to G$, $(x, y) \mapsto xy$, written multiplicatively, such that the following axioms hold:

- associativity: $x(yz) = (xy)z$ for all $x, y, z \in G$;
- identity: there is an identity element $1 \in G$ satisfying $1x = x1 = x$ for all $x \in G$;
- inverse: each element $x \in G$ has an inverse x^{-1} satisfying $xx^{-1} = x^{-1}x = 1$.

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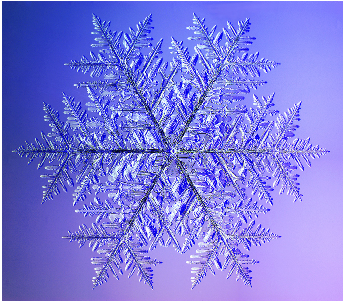
But what is group theory really all about?

What is group theory?

- Group theory is the mathematical study of symmetry.
- The axioms of a group formalize the essential aspects of symmetry.



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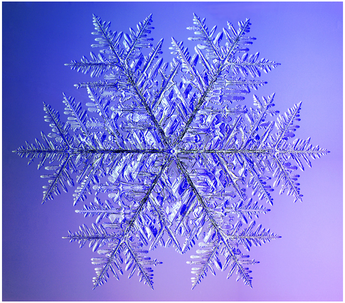


Symmetry (Encyclopaedia Britannica)

"In geometry, the property by which the sides of a figure or object reflect each other across a line (axis of symmetry) or surface; in biology, the orderly repetition of parts of an animal or plant; in chemistry, a fundamental property of orderly arrangements of atoms in molecules or crystals; in physics, a concept of balance illustrated by such fundamental laws as the third of Newton's laws of motion."

Symmetry groups

The symmetries of any (mathematical) object form a group (called its automorphism group).

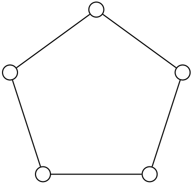
"Whatever you have to do with a structure-endowed entity Σ try to determine its group of automorphisms... You can expect to gain a deep insight into the constitution of Σ in this way."

Hermann Weyl, *Symmetry (1952)*.

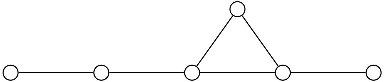
Graphs and symmetry

Definition

A graph Γ consists of a set $V\Gamma$ of vertices, and set $E\Gamma$ of edges (unordered pairs of distinct vertices).



Γ_1



Γ_2

Graphs and symmetry

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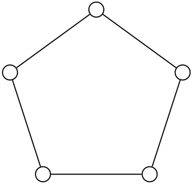

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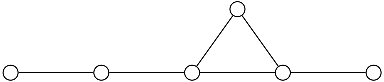

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- Γ_1 has "more symmetry" than Γ_2.

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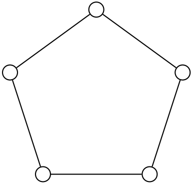

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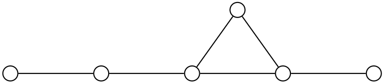


Γ_2

- Γ_1 has "more symmetry" than Γ_2.
- Imagine you are trapped inside the graph:
  - In Γ_1 the world looks the same from every vertex.
  - In Γ_2 the world looks different from each vertex.

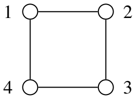
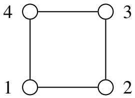
Automorphisms

- An isomorphism $\phi : \Gamma_1 \to \Gamma_2$ of graphs is a bijection that maps edges to edges, and non-edges to non-edges.
- An automorphism of a graph Γ is an isomorphism $\phi : \Gamma \to \Gamma$.
- $\mathrm{Aut}(\Gamma)$ - the full automorphism group of the graph Γ.

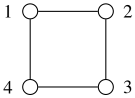
Automorphisms

Γ - graph, $\quad \mathrm{Aut}(\Gamma)$ - automorphism group

Example (Γ - a square)


$\mapsto$


$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \in \mathrm{Aut}(\Gamma)$$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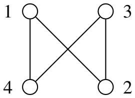
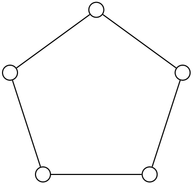


$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \notin \mathrm{Aut}(\Gamma)$$

Here $\mathrm{Aut}(\Gamma)$ is the dihedral group of all 8 symmetries of the square (4 reflections & 4 rotations).

Automorphisms and symmetry



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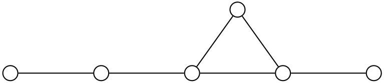

Γ_2

- For every pair of vertices w, v of Γ_1 there is an automorphism of Γ_1 that maps w to v.
  - $\text{Aut}(\Gamma_1)$ = rotations + reflections (dihedral group).
- The only automorphism of Γ_2 is the identity mapping:
  - i.e. $|\text{Aut}(\Gamma_2)| = 1$.

Frucht's theorem

Theorem (Frucht (1938))

Every group is isomorphic to the automorphism group of some graph.

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Every group is isomorphic to the automorphism group of some graph.

Conclusion: The study of automorphism groups of graphs is as difficult as the study of groups in general.

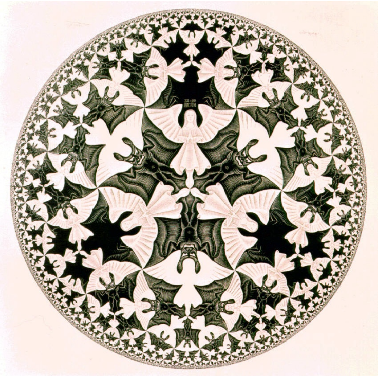
Idea:

- Identify nice / natural symmetry conditions on graphs.
- Study the resulting families of graphs and their automorphism groups.

Local or global?

Among other definitions of symmetry, the dictionary will often list the following two:

- exact correspondence of parts;
- remaining unchanged by transformation.



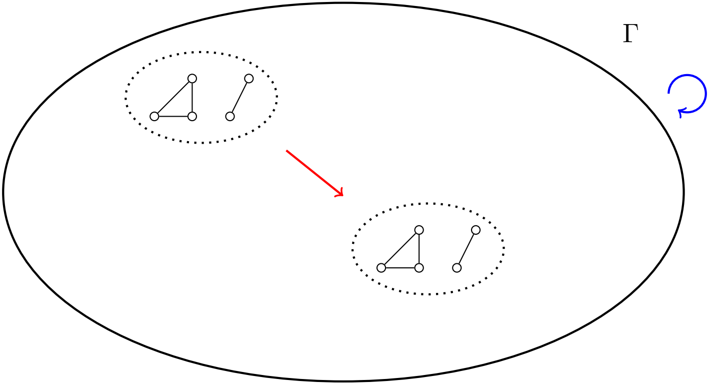
Mathematicians usually consider the second, global, notion, but what about the first, local, notion, and what about the relationship between them?

Homogeneous graphs

Definition

A graph Γ is homogeneous if every isomorphism between finite induced subgraphs of Γ can be extended to an automorphism of Γ.

homogeneous $\equiv$ any local symmetry is global



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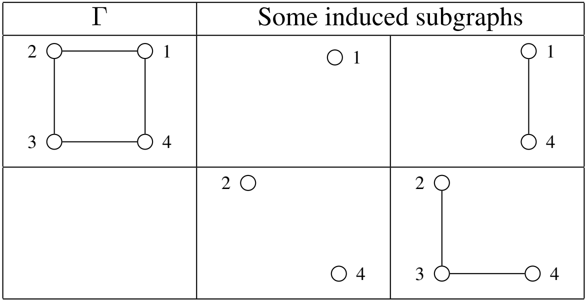
Induced subgraphs.



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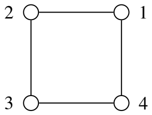
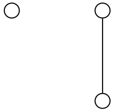
Induced subgraphs.

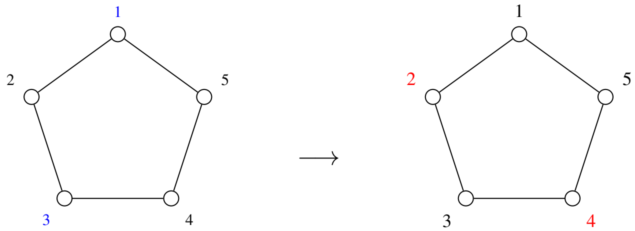
However, the graph Γ:



has no induced subgraph isomorphic to:



Extending isomorphisms to automorphisms

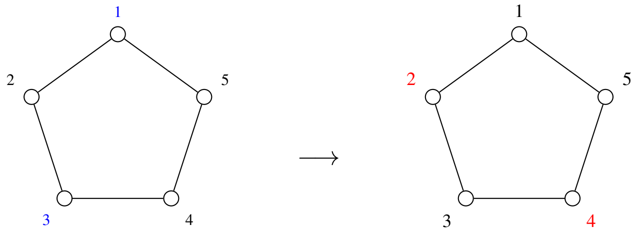


The isomorphism

$$(1, 3) \mapsto (4, 2)$$

between finite induced subgraphs

Extending isomorphisms to automorphisms



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$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 & 5 \end{pmatrix}$$

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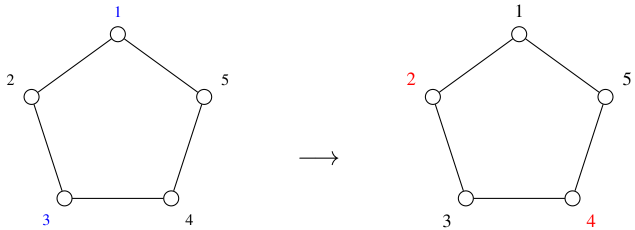

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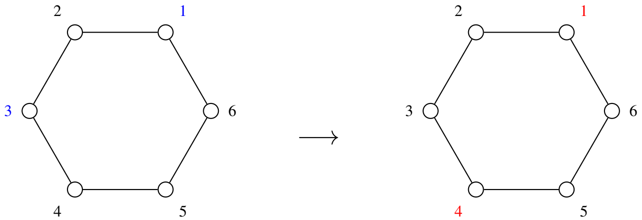
Fact. The pentagon is a homogeneous graph.

Extending isomorphisms to automorphisms



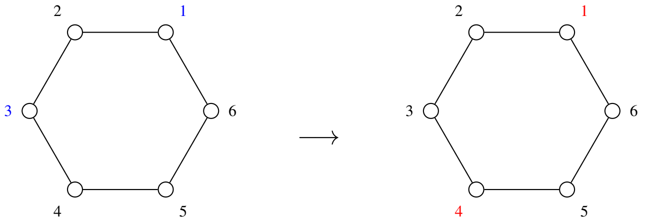
The isomorphism between finite induced subgraphs

$$(1, 3) \mapsto (1, 4)$$

does not extend to the automorphism.

e.g. There is a path of length two from 1 to 3, while there is no path of length two from 1 to 4.

Extending isomorphisms to automorphisms



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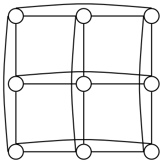
So the hexagon is not a homogeneous graph.

The finite homogeneous graphs

Theorem (Gardiner (1976))

A finite graph is homogeneous if and only if it is isomorphic to one of the following (or its complement):

1. *finitely many disjoint copies of a complete graph K_r (graph with r vertices where every pair is joined by an edge)*
2. *the pentagon C_5*
3. *the graph $K_3 \times K_3$ drawn below*

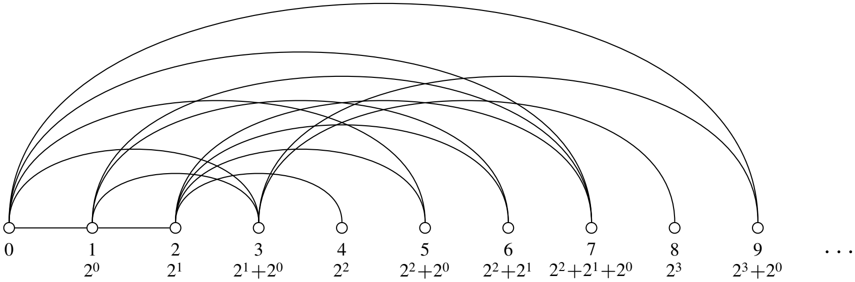


The random graph

Definition

Constructed by Rado (1964). The vertex set is the natural numbers $\mathbb{N}_0$ (including zero).

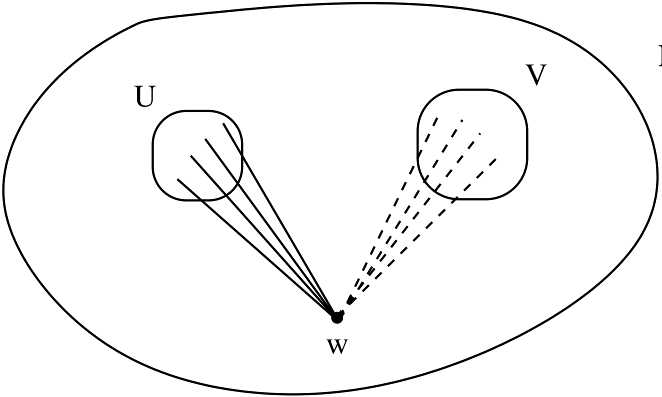
For $i, j \in \mathbb{N}_0$, $i < j$, then i and j are joined if and only if the ith digit in j in base 2, reading right-to-left, is 1.



The random graph

Consider the following property of graphs:

(*) For any two finite disjoint sets U and V of vertices, there exists a vertex w, not in $U \cup V$, adjacent to every vertex in U and to no vertex in V.



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Theorem

There exists a countably infinite graph R satisfying property (*), and it is unique up to isomorphism. The graph R is homogeneous.

Existence. The random graph R defined above satisfies property (*).

Uniqueness and homogeneity. Both follow from a back-and-forth argument. Property (*) is used to extend the domain (or range) of any isomorphism between finite substructures one vertex at a time.

Existence

Claim. The random graph has property (*).

Proof (by example).
Challenge - Find a vertex adjacent to all of

$$U = \{2, 19, 257\}$$

but not adjacent to any of

$$V = \{0, 3, 36, 1006\}.$$

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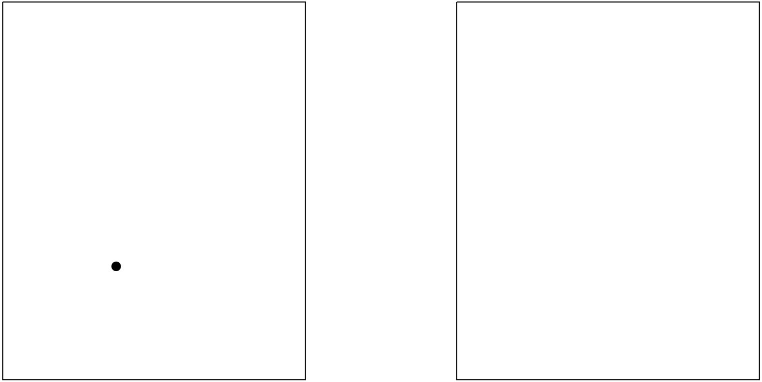
Solution - For instance, you can take the vertex:

$$n = 2^{2} + 2^{19} + 2^{257} + 2^{1000000} \in \mathbb{N}.$$

Uniqueness and homogeneity

Back-and-forth argument

Claim. (i) Any two countable graphs with property (*) are isomorphic.



Γ_1 Γ_2

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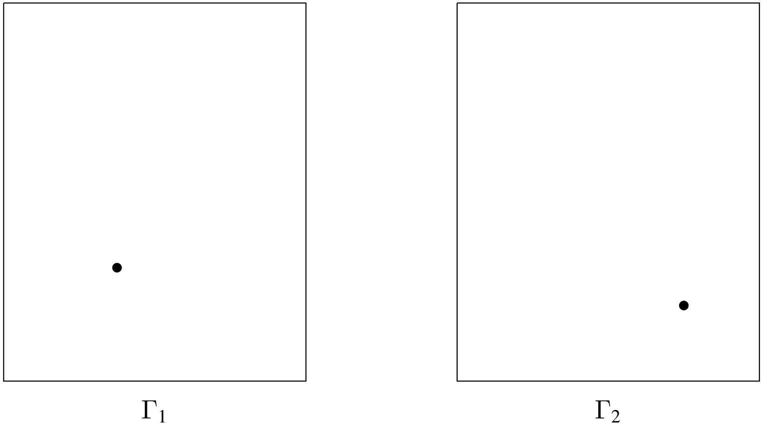

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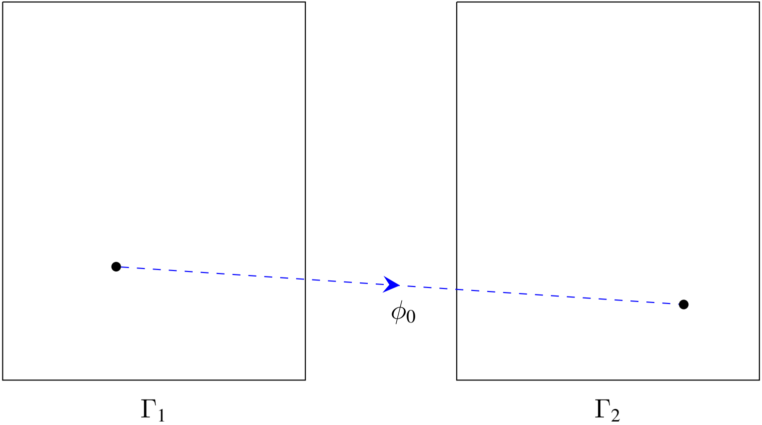

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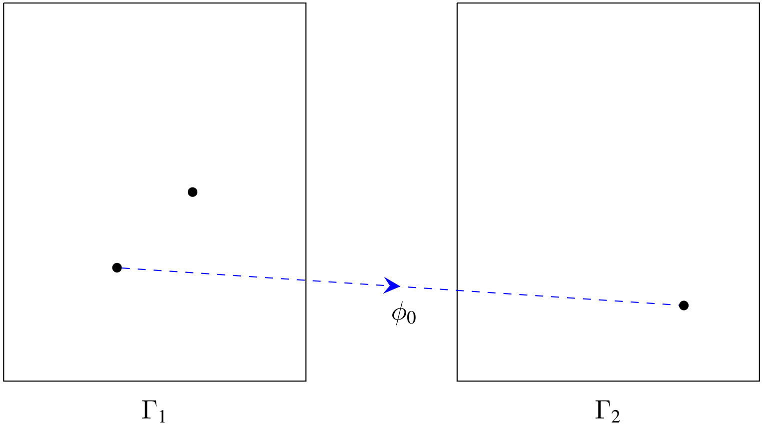

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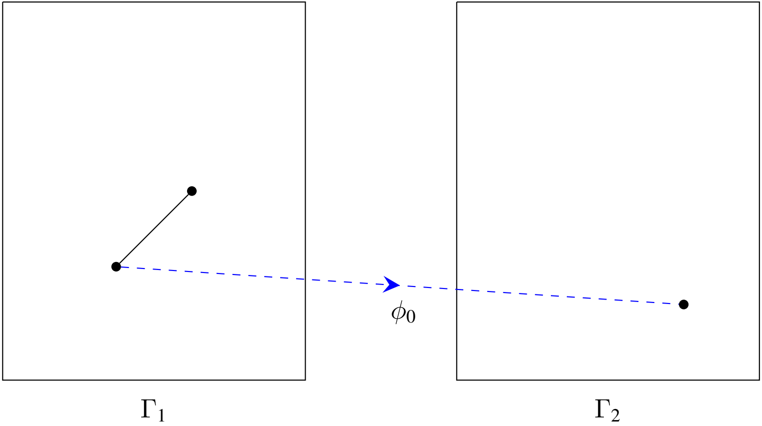

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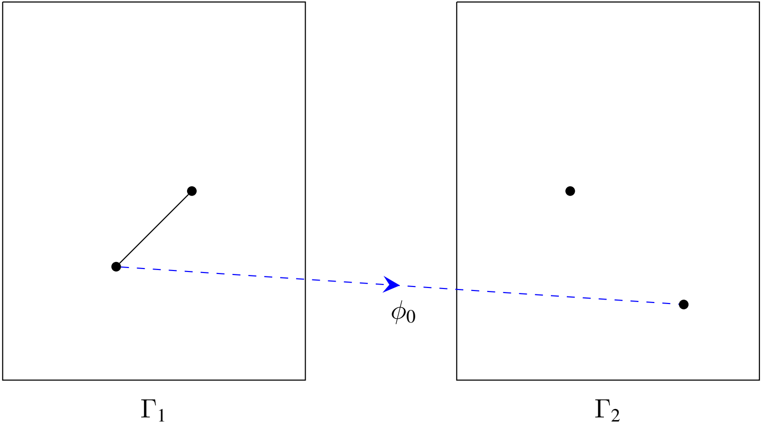

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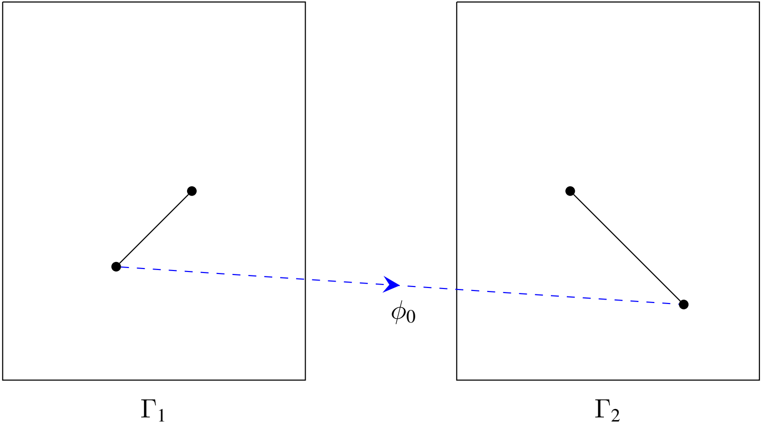

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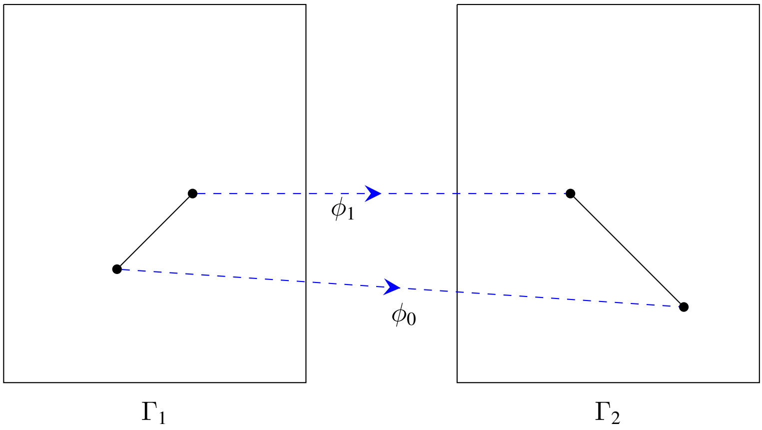

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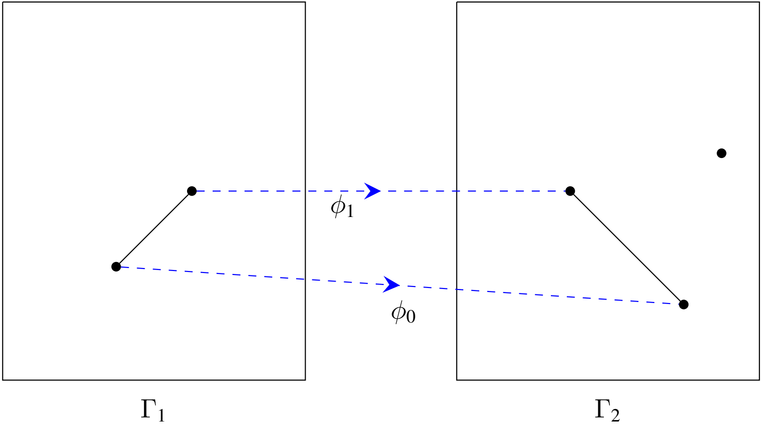

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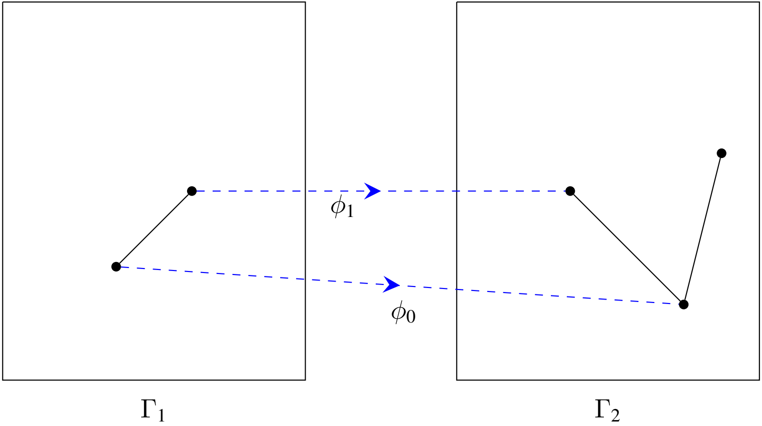

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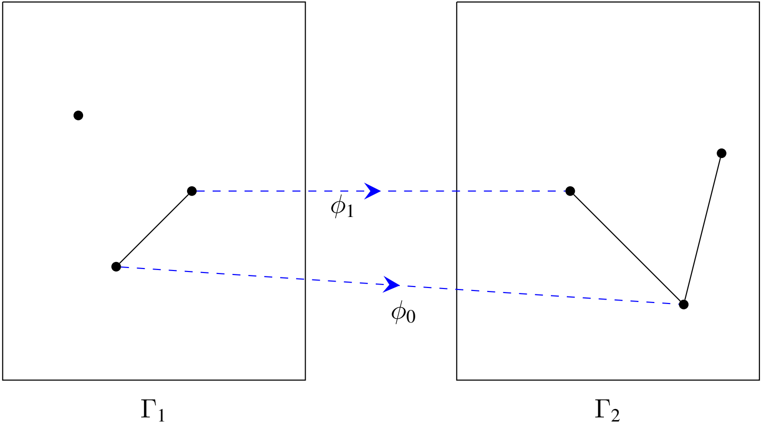

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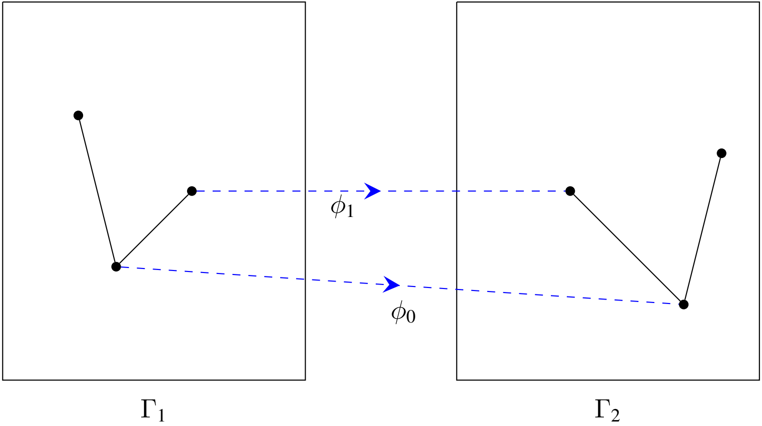

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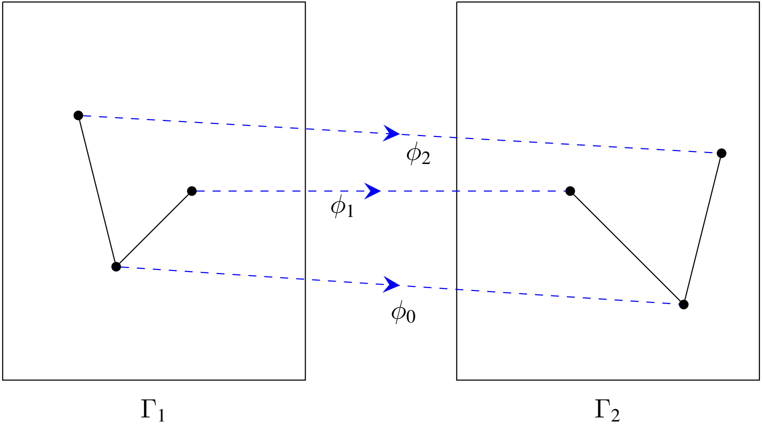

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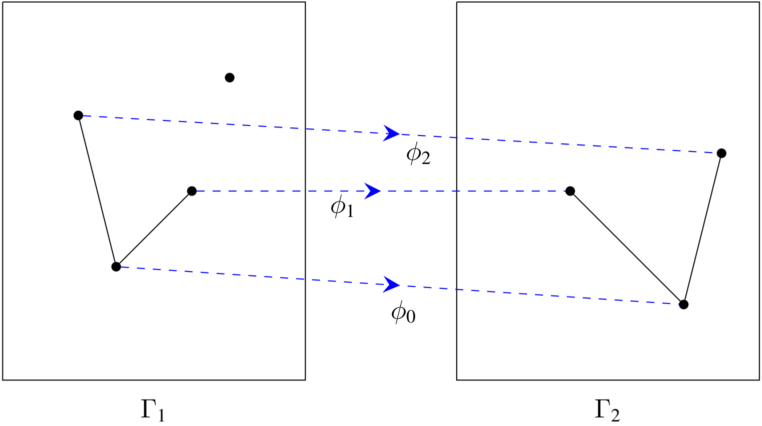

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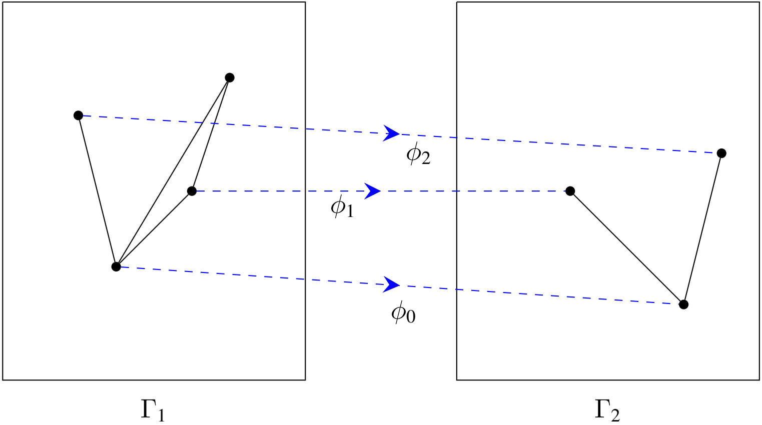

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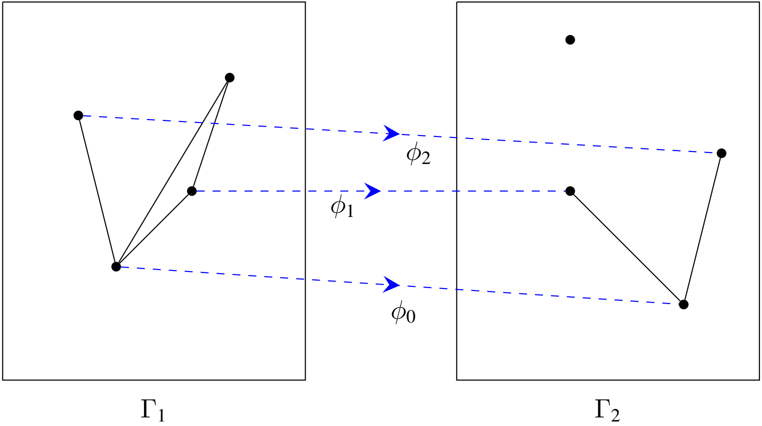

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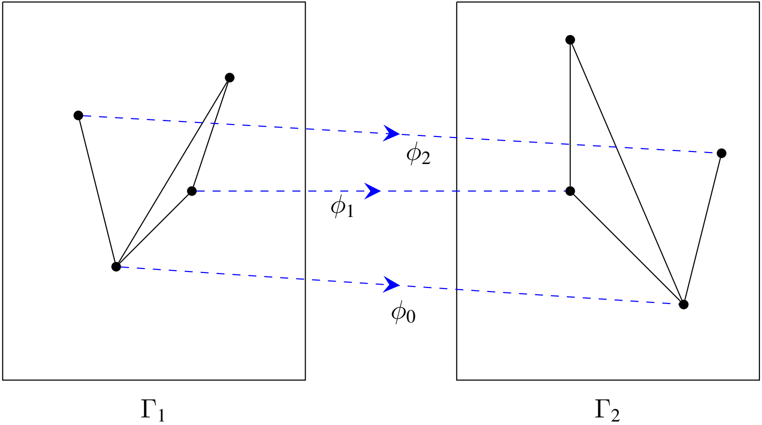

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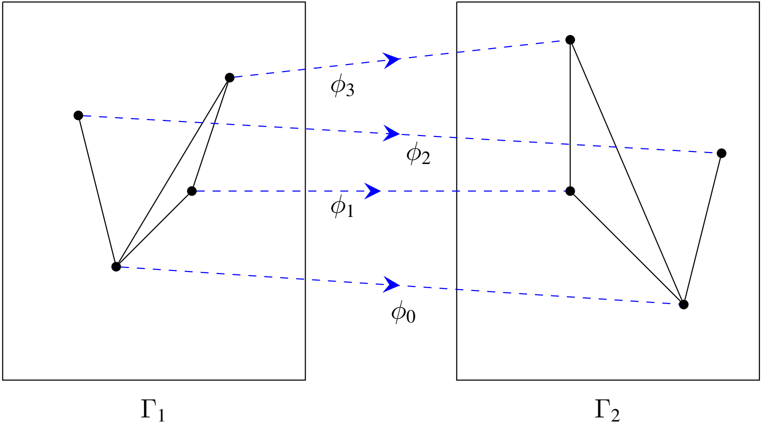

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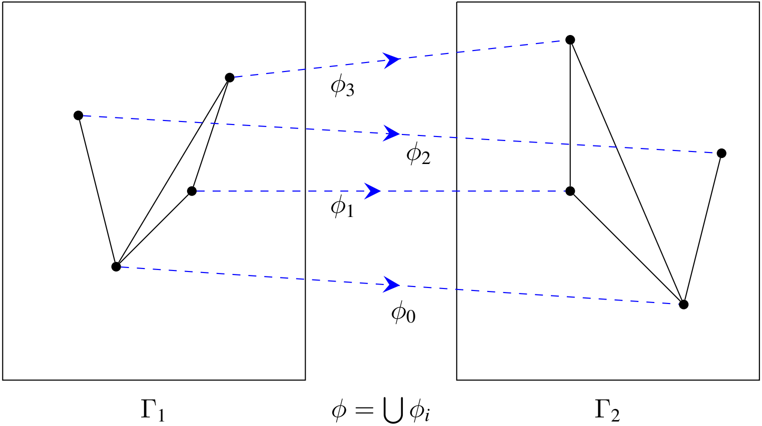
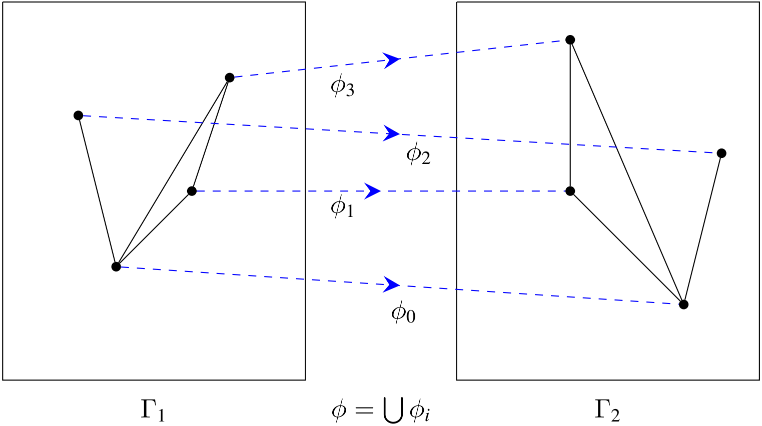


Uniqueness and homogeneity

Back-and-forth argument

Claim. (i) Any two countable graphs with property (*) are isomorphic.
(ii) Any countable graph with property (*) is homogeneous.



The random graph

Consider the following property of graphs:

(*) For any two finite disjoint sets U and V of vertices, there exists a vertex w, not in $U \cup V$, adjacent to every vertex in U and to no vertex in V.

Theorem

There exists a countably infinite graph R satisfying property (*), and it is unique up to isomorphism. The graph R is homogeneous.

Existence. The random graph R defined above satisfies property (*).

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Properties of the random graph

The random graph R has the following properties:

- Random: if we choose a countable graph at random (edges independently with probability $\frac{1}{2}$), then with probability 1 it is isomorphic to R (Erdös and Rényi, 1963).

Properties of the random graph

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- Universal: it embeds every countable graph as an induced subgraph.
- Partition property: for any partition $V\Gamma = X \cup Y$ either the subgraph induced by X, or the subgraph induced by Y, is again isomorphic to R.
  - Aside from the complete graph, and empty graph, R is the only countable graph with this property.

Automorphisms of the random graph

The automorphism group of R is a very interesting group. It has the following properties:

- $|\operatorname{Aut}(R)| = 2^{\aleph_0}$ - it is an uncountably infinite group.
- $\operatorname{Aut}(R)$ is simple (Truss, 1985)
  - i.e. it contains no proper normal subgroups.
- $\operatorname{Aut}(R)$ contains a copy of every finite or countable group as a subgroup i.e. it is a universal group.

Homogeneous structures and Fraïssé's theorem

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- The age of a graph Γ is the set of all finite induced subgraphs of Γ.

Fraïssé:

- showed that a countable homogeneous graph is uniquely determined by its age;
- described how a homogeneous graph is "built up" from its finite induced subgraphs.

Example

R - the random graph, $\quad \mathrm{Age}(R) = \{\text{all finite graphs}\}$

R is the unique countable homogeneous graph with this age.

Countable homogeneous graphs

Examples (Henson (1971))

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Theorem (Lachlan and Woodrow (1980))

Let Γ be a countably infinite homogeneous graph. Then Γ (or its complement) is isomorphic to one of: a disjoint union of complete graphs, the random graph, or a Henson graph H_n for some $n \geq 3$.

Other structures

Fraïssé's theory applies to mathematical structures generally, not just to graphs, giving rise to a whole host of other interesting infinite homogeneous structures, including:

- the random tournament, directed graph, hypergraph, etc.
- the universal homogeneous total order $\mathbb{Q}$ (Cantor, 1895), partial order, etc.
- the universal locally finite group (Hall, 1959).

The study of homogeneous structures, and their automorphism groups, is still a very active area of research, with many questions still unanswered.

Some open problems

Henson graphs

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Metric spaces

A connected graph is easily seen to be a metric space (i.e. a space $X = (X, d)$ where d is a distance function) with the distance between vertices being the length of a shortest path connecting them.

- Classify the countable homogeneous metric spaces.