Homotopical and homological finiteness properties of monoids and their subgroups

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Presentations

Fact: There are lots of *nasty* finitely presented monoids out there.

Markov (1947), Post (1947): There exist finitely presented monoids for which there is no algorithm to solve the word problem.

Idea

1. Identify a class $\mathcal{C}$ of "nice" finite presentations:
    - finite complete rewriting systems
        - = noetherian and confluent

    A monoid defined my a finite complete rewriting system has solvable word problem.
2. Try to gain understanding of those monoids that may be defined by presentations from $\mathcal{C}$:
    - study properties of monoids defined by such rewriting systems:
        - Finite derivation type (FDT)
        - FP_n

Finite derivation type
a homotopical finiteness condition

- Is a property of finitely presented monoids.
- Introduced by **Squier (1994)**
  - (and independently by Pride (1995))

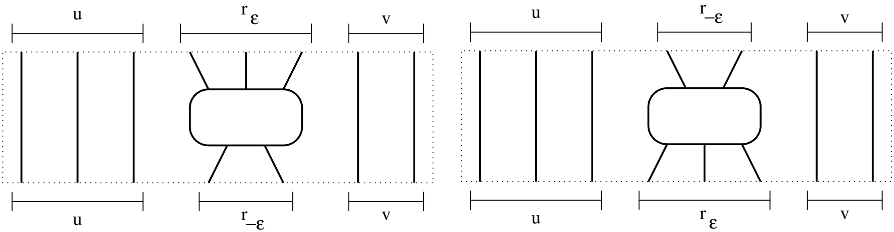
Original motivation

To capture much of the information of a finite complete rewriting system for a monoid in a property which is independent of the choice of presentation.

- Connections with diagram groups (which are fundamental groups of Squier complexes of monoid presentations)
  - **Kilibarda (1997)**
  - **Guba & Sapir (1997)**

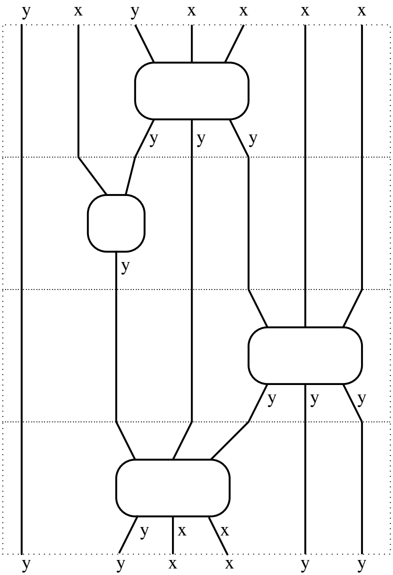
The derivation graph of a presentation

- $\mathcal{P} = \langle A | R \rangle$ a monoid presentation
  - A - alphabet, $R \subseteq A^* \times A^*$ - rewrite rules
- Derivation graph: $\Gamma = \Gamma(\mathcal{P}) = (V, E, \iota, \tau, ^{-1})$:
  - Vertices: $V = A^*$
  - Edges are 4-tuples: $\{(u, r, \epsilon, v) : u, v \in A^*, r = (r_{+1}, r_{-1}) \in R, \text{ and } \epsilon \in \{+1, -1\}\}$.
- Initial and terminal vertices: $\iota, \tau : E \to V$ for $\mathbb{E} = (u, r, \epsilon, v)$:
  - $\iota\mathbb{E} = ur_\epsilon v$, $\tau\mathbb{E} = ur_{-\epsilon}v$
- Inverse edge mapping: $^{-1} : E \to E$
  - $(u, r, \epsilon, v)^{-1} = (u, r, -\epsilon, v)$.



$\mathbb{E} = (u, r, \epsilon, v)$ $\mathbb{E}^{-1} = (u, r, -\epsilon, v)$

Paths and pictures



Example. $\langle x, y | \underbrace{xy = y}_{r}, \underbrace{yx^2 = y^3}_{s} \rangle$

A path is a sequence $\mathbb{P} = \mathbb{E}_1 \circ \mathbb{E}_2 \circ \ldots \circ \mathbb{E}_n$ where $\tau\mathbb{E}_i \equiv \iota\mathbb{E}_{i+1}$.

Gluing edge-pictures together we obtain pictures for paths.

ι and τ can be defined for paths

In this example
$\iota\mathbb{P} = yxyxxxx, \;\; \tau\mathbb{P} = yyxxyy.$

Operations on pictures

$\mathcal{P} = \langle A | R \rangle, \quad \Gamma = \Gamma(\mathcal{P})$

$$\text{Pictures} \rightsquigarrow \text{Paths}$$

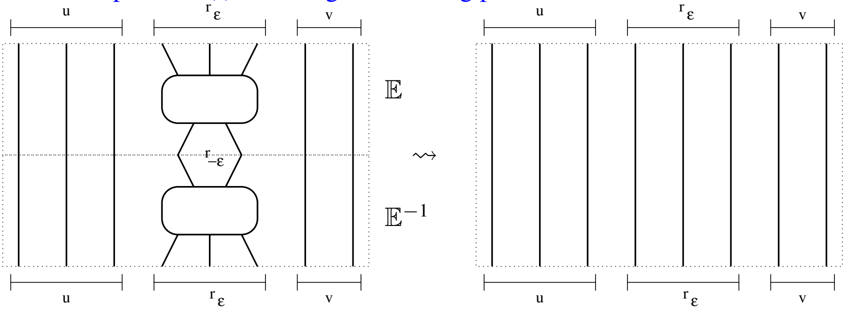
- Parallel paths: write $\mathbb{P} \parallel \mathbb{Q}$ if $\iota\mathbb{P} \equiv \iota\mathbb{Q}$ and $\tau\mathbb{P} \equiv \tau\mathbb{Q}$.
- **X** - set of pairs of paths $(\mathbb{P}_1, \mathbb{P}_2)$ such that $\mathbb{P}_1 \parallel \mathbb{P}_1$.

Idea

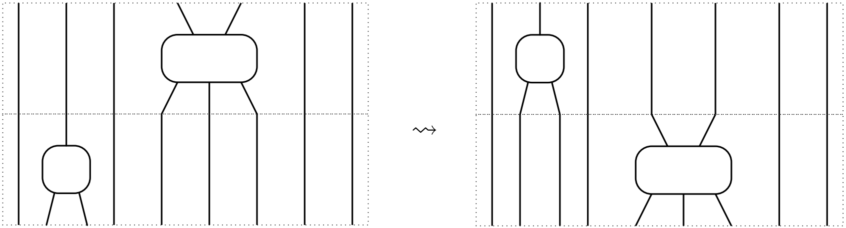
Want to regard certain paths as being equivalent to one another modulo **X**.

Operations on pictures

Basic operation (I): Deleting a cancelling pair



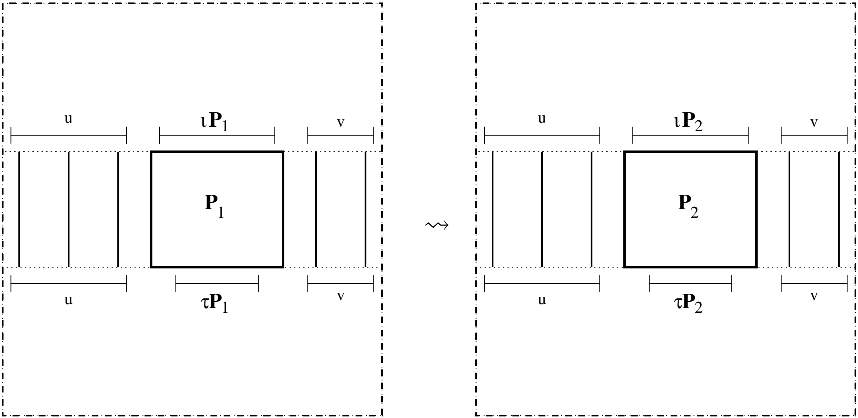
Basic operation (II): Interchanging disjoint discs



Operations on pictures

Basic operation (III): Replacing a subpicture using **X**

Replace a subpicture $\mathbb{P}_1$ by $\mathbb{P}_2$ provided $(\mathbb{P}_1, \mathbb{P}_2) \in \mathbf{X}$.



$$u \cdot \mathbb{P}_1 \cdot v \quad \rightsquigarrow \quad u \cdot \mathbb{P}_2 \cdot v$$

Homotopy bases

Note: Applications of these picture operations do not change the initial vertex or the terminal vertex of the original path.

A homotopy base is...

a set $\mathbf{X}$ of parallel paths such that given an arbitrary pair $(\mathbb{P}_1, \mathbb{P}_2) \in \|$ we can transform $\mathbb{P}_1$ into $\mathbb{P}_2$ by a finite sequence of elementary picture operations (and their inverses)

(I) cancelling pairs, (II) disjoint discs, (III) applying $\mathbf{X}$.

Finite derivation type

Definition

$\mathcal{P} = \langle A|R \rangle$ has finite derivation type (FDT) if there is a **finite homotopy base** for $\Gamma = \Gamma(\mathcal{P})$. A monoid M has FDT if it may be defined by a presentation with FDT.

Theorem (Squier (1994))

- *The property FDT is independent of choice of finite presentation.*
- *Let M be a finitely presented monoid. If M has a presentation by a finite complete rewriting system then M has FDT.*

Monoids and their subgroups

Idea

Relate the problem of understanding a property for monoids with the problem of understanding the property for groups.

- M - monoid
- Green's relations $\mathcal{R}$, $\mathcal{L}$, and $\mathcal{H}$

$$x\mathcal{R}y \Leftrightarrow xM = yM, \;\; x\mathcal{L}y \Leftrightarrow Mx = My, \;\; \mathcal{H} = \mathcal{R} \cap \mathcal{L}.$$

- H = an $\mathcal{H}$-class. If H contains an idempotent e then H is a group with identity e.
  - These are precisely the maximal subgroups of M.

General question: How do the properties of M relate to those of the maximal subgroups of M?

Finite derivation type for subgroups of monoids

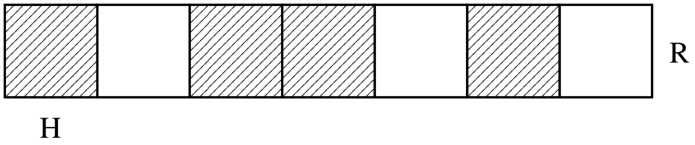
(joint work with A. Malheiro)

Theorem

Let M be a monoid and let H be a maximal subgroup of M. If the $\mathcal{R}$-class of H contains only finitely many $\mathcal{H}$-classes then:

- *M has FDT $\Rightarrow$ H has FDT.*

- Given a homotopy base $\mathbf{X}$ for M we show how to construct a homotopy base $\mathbf{Y}$ for H. Finiteness is preserved when the $\mathcal{R}$-class has only finitely many $\mathcal{H}$-classes.
- **Ruskuc (1999):** Proved the corresponding result for finite presentability.



Regular monoids

- A semigroup is regular if every $\mathcal{R}$-class (equivalently every $\mathcal{L}$-class) contains an idempotent.

Theorem

Let M be a regular monoid with finitely many left and right ideals. Then M has finite derivation type if and only if every maximal subgroup of M has finite derivation type.

Notes on proof. We show in general how to construct a homotopy base for M from homotopy bases of the maximal subgroups.

Complete rewriting systems

Theorem

Let M be a regular monoid with finitely many left and right ideals. If every maximal subgroup of M has a presentation by a finite complete rewriting system then so does M.

- The converse is still open.
- This relates to the following open problem from group theory:

  Question. Is the property of having a finite complete rewriting system preserved when taking finite index subgroups?

The finiteness condition FP_n

- **Wall (1965):** introduced a (geometric) finiteness condition for groups called $\mathcal{F}_n$:
  - $\mathcal{F}_1 \equiv$ finite generation
  - $\mathcal{F}_2 \equiv$ finite presentability
- Issue: $\mathcal{F}_n$ not very tractable in terms of using algebraic machinery
- **Bieri (1976):** introduced FP_n for groups.

Definition

A monoid M is of type left-FP_n if there is a resolution:

$$F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

of the trivial left $\mathbb{Z}M$-module $\mathbb{Z}$ such that $F_0, F_1, \ldots, F_n$ are finitely generated free left $\mathbb{Z}M$-modules. A monoid is of type left-FP_∞ if it is left-FP_n for all $n \in \mathbb{N}$.

FP_n and FDT

- **Kobayashi (1990):**

M presented by a finite complete rewriting system	$\Rightarrow$	M is of type left-FP_∞

- **Cremanns & Otto (1994) / Lafont (1995) / Pride (1995):** For finitely presented monoids

$$\mathrm{FDT} \Rightarrow \mathrm{FP}_3.$$

- **Cremanns & Otto (1996):** for finitely presented groups

$$\mathrm{FDT} \equiv \mathrm{FP}_3.$$

Corollary (of our FDT results)

Let M be a finitely presented regular monoid with finitely many left and right ideals. If every maximal subgroup of M is of type FP_3 *then M is of type left*-FP_3.

FP_n and maximal subgroups

(joint work with S. J. Pride)

Definition

A semigroup is simple if it has no proper ideals.

Theorem

Let S be a simple semigroup with finitely many left and right ideals. Then the monoid S^1 is of type left-FP_n if and only if all of its maximal subgroups are of type FP_n.

(Of course, all the maximal subgroups are isomorphic here.)

FP_n for monoids with zero

Proposition (Kobayashi (preprint))

If a monoid M has a zero element then M is if type left-FP_∞

Example

G - any group, $M = G^0$ - adjoin a zero ($0g = g0 = 00 = 0$).

- Maximal subgroups of M are: $H_1 = G$, and $H_0 = \{0\}$.
- Kobayashi $\Rightarrow$ M is left-FP_∞.
- G can have any properties we like
  - e.g. can choose G not to be of type FP_n for any given n.

FP_n and maximal subgroups
minimal ideals

Theorem

Let M be a monoid that has a minimal ideal G which is a group. Then M is of type left-FP_n if and only if G is of type FP_n.

Definition

Clifford monoid - a regular monoid whose idempotents are central

Theorem

A Clifford monoid is of type left-FP_n if and only if it has a minimal ideal G (which is necessarily a group) and G is of type FP_n.

Combining the two results

- For FP_1 we have:

Theorem

Let S be a monoid with a minimal ideal J such that J has finitely many left and right ideals. Let G be a maximal subgroup of J. Then S is of type left-FP_1 if and only if G is of type FP_1.

Corollary

Let S be a monoid with finitely many left and right ideals. Let G be a maximal subgroup of the unique minimal ideal of S. Then S is of type left-FP_1 if and only if G is of type FP_1.

- Currently in the process of extending this to left-FP_n ($n \geq 2$).
- **For the future:** What about monoids without minimal ideals?