The word problem, divisibility problem, and conjugacy problem for one-relator monoids

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Equations over free monoids

- $A = \{a, b, \ldots\}$ - alphabet, $\Omega = \{X, Y, \ldots\}$ - set of variables,
- Word equation: a pair $(L, R) \in (A \cup \Omega)^* \times (A \cup \Omega)^*$ written $L = R$.
- System of word equations: $\{L_1 = R_1, \ldots, L_k = R_k\}$.
- Solution: a homomorphism $\sigma : (A \cup \Omega)^* \to A^*$ leaving A invariant such that $\sigma(L_i) = \sigma(R_i)$ for $1 \leqslant i \leqslant k$.

Example

$A = \{a, b\}, \Omega = \{X, Y, Z, U\}$

$$XaUZaU = YZbXaabY$$

One solution is given by σ defined by

$$X \mapsto abb,\ Y \mapsto ab,\ Z \mapsto ba,\ U \mapsto bab.$$

Theorem (Makanin (1977)). There is an algorithm which decides whether a system of equations over the free monoid has a solution.

Equations over finitely presented monoids

$$\langle A \mid R \rangle = \langle \underbrace{a_1, \ \ldots, \ a_n}_{\text{generators}} \quad \mid \quad \underbrace{u_1 = v_1, \ \ldots, \ u_m = v_m}_{\text{defining relations}} \rangle$$

- Defines $M = A^*/\rho$ where ρ is the smallest congruence on A^* containing R.
- For $w \in A^*$ we use $[w]$ to denote the element $w/\rho \in M$.
- Solution to a system of equations $\{L_1 = R_1, \ldots, L_k = R_k\}$: a homomorphism $\sigma : (A \cup \Omega)^* \to A^*$ leaving A invariant such that $[\sigma(L_i)] = [\sigma(R_i)]$ for $1 \leqslant i \leqslant k$.

Fact: If there is an algorithm for solving equations in $\langle A \mid R \rangle$ then M must have decidable word problem.

Solving equations in one-relator monoids

Longstanding open problem

Is the word problem decidable for one-relator monoids $\langle A \mid u = v \rangle$?

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Question

In which examples / classes of one-relator monoids is solvability of equations decidable?

Natural classes one might consider:

- $\langle A \mid u = v \rangle$ where $|u| = |v|$ - homogeneous presentations.
- $\langle A \mid u = v \rangle$ where u and v have distinct initial letters and distinct terminal letters $\Rightarrow$ monoid is group embeddable.
- $\langle A \mid w = 1 \rangle$ the so-called 'special' one-relator monoids.

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- $\langle A \mid w = 1 \rangle$ the so-called 'special' one-relator monoids.

Question: Is there an algorithm for solving equations in the bicyclic monoid $\langle b, c \mid bc = 1 \rangle$?

Word problem and divisibility problem in $\langle A \mid w = 1 \rangle$

Word problem
Setting $\Omega = \varnothing$, for $u, v \in A^*$ we are asking whether $u = v$ has a solution.

Theorem (Adjan 1966)

The word problem is decidable for special one relator monoids $\langle A \mid w = 1 \rangle$.

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Divisibility problem
For two words $u, v \in A^*$ we say u is left divisible by v if there is a word $z \in A^*$ such that $[u] = [vz]$.

Setting $\Omega = \{X\}$ we are asking whether the equation

$$u = vX$$

has a solution.

Theorem (Makanin 1966)

The left divisibility problem is decidable for special one relator monoids $\langle A \mid w = 1 \rangle$.

Conjugacy in $\langle A \mid w = 1 \rangle$

Left conjugacy

Set $\Omega = \{X\}$. The words $u, v \in A^*$ are left conjugate if the equation

$$uX = Xv$$

has a solution.

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Set $\Omega = \{X, Y\}$. The words $u, v \in A^*$ are cyclically conjugate if the system of equations

$$\{u = XY,\ v = YX\}$$

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Theorem (Otto 1984 & Zhang 1991)

In $\langle A \mid w = 1 \rangle$ two words are left conjugate if and only if they are cyclically conjugate. These define equivalence relations on the monoid.

The conjugacy problem in $\langle A \mid w = 1 \rangle$

Theorem (Zhang 1989)

Let M be the monoid defined by $\langle A \mid w = 1 \rangle$ and let G be the group of units of M. If G has decidable conjugacy problem then M has decidable conjugacy problem.

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Corollary (Zhang 1989)

The one relator monoids $\langle A \mid u^n = 1 \rangle$, with $n > 1$, have decidable conjugacy problem.

Proof. Let M the monoid defined by this presentation. By Adjan (1966) G is a one-relator group with torsion. It follows my Newman (1968) that G has decidable conjugacy problem, and hence so does M. $\square$

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Problem: Is solvability of equations decidable in the one relator monoids $\langle A \mid u^n = 1 \rangle$, with $n > 1$?

The group of units of $M \cong \langle A \mid w = 1 \rangle$

Construct a sequence

$$\{w\} = C_1 \subseteq C_2 \subseteq C_3 \subseteq \ldots \subseteq C_k \subseteq C_{k+1} \subseteq \ldots$$

$$C_{i+1} = C_i \cup \{xy \mid x \in W(C_i) \ \& \ yx \in C_i\} \cup \{zx \mid x \in W(C_i) \ \& \ xz \in C_i\}$$

$$W(C_i) = \{x \in A^+ : x \text{ is a prefix of some word from } C_i \text{ and } x \text{ is a suffix of some word from } C_i\}$$

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$\Delta = \{u \in W(C_k) : \text{no proper prefix of } u \text{ is in } W(C_k)\}$

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Δ is a prefix code $\Rightarrow$ this extends to a homomorphism $\phi : \Delta^* \to B^*$.

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Δ is a prefix code $\Rightarrow$ this extends to a homomorphism $\phi : \Delta^* \to B^*$.

Theorem (Adjan 1966). The group of units G of M is defined by the presentation $\langle B \mid \phi(w) = 1 \rangle$.

Noetherian confluent rewriting systems

A - alphabet, $R \subseteq A^* \times A^*$ - rewrite rules, $\langle A \mid R \rangle$ - rewriting system
Write $r = (r_{+1}, r_{-1}) \in R$ as $r_{+1} \to r_{-1}$.

Define a binary relation $\to_R$ on A^* by

$$u \to_R v \;\Leftrightarrow\; u = w_1 r_{+1} w_2 \text{ and } v = w_1 r_{-1} w_2$$

for some $(r_{+1}, r_{-1}) \in R$ and $w_1, w_2 \in A^*$.

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$\overset{*}{\underset{R}{\longrightarrow}}$ is the transitive and reflexive closure of $\underset{R}{\to}$.
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Noetherian: No infinite descending chain

$$w_1 \to_R w_2 \to_R \cdots \to_R w_n \to_R \cdots$$

Confluent: Whenever

$$u \xrightarrow{*}_R v \text{ and } u \xrightarrow{*}_R v'$$

there is a word $w \in A^*$:

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Normal forms: If $\langle A \mid R \rangle$ is noetherian and confluent then each $\xleftrightarrow{*}_R$ -class contains a unique word which is irreducible with respect to $\to_R$.

Zhang's method

$M \cong \langle A \mid w = 1 \rangle, \quad G \cong \langle B \mid \phi(w) = 1 \rangle$ where $\phi : \Delta^* \to B^*$.

Zhang's infinite rewriting system

Shortlex order: For $x, y \in A^*$ write $x < y$ if $|x| < |y|$ or $|x| = |y|$ and $x <_{\text{lex}} y$.

Theorem (Zhang 1992)

Let $R = \{(w, 1)\}$ and

$$S = \{(u, v) : u, v \in \Delta^*,\ v < u \text{ and } \phi(u) = \phi(v) \text{ in } G\}.$$

Then S is a noetherian and confluent, and $\overset{*}{\leftrightarrow}_R \ = \overset{*}{\leftrightarrow}_S$.

That is, the presentation $\langle A \mid w = 1 \rangle$ is equivalent to the noetherian confluent presentation $\langle A \mid S \rangle$.

Computing normal forms

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Theorem (Zhang 1992)

$S = \{(u, v) : u, v \in \Delta^*, \; u > v$ and $\phi(u) = \phi(v)$ in $G\}$.
$\langle A \mid w = 1 \rangle$ is equivalent to the noetherian confluent presentation $\langle A \mid S \rangle$.

We call $x \in A^*$ irreducible if no rewrite rule $\to_S$ can be applied to it. We use $\bar{x}$ to denote the unique irreducible word equal to x in M and call $\bar{x}$ the normal form of x.

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- List the pairs $(u_1, v_1), (u_2, v_2), \ldots$ with $u_i, v_i \in \Delta^*$, $u_i > v_i$ and such that $|u_i| \leqslant |x|$ and $\phi(u_i) = \phi(v_i)$ in G.

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- For each pair (u_i, v_i) check whether this relation can be applied to x.

Word problem using Zhang normal form

Theorem (Adjan 1966, Zhang 1992)

The word problem is decidable for special one relator monoids $\langle A \mid w = 1 \rangle$.

Proof: Let M be the special one-relator monoid $\langle A \mid w = 1 \rangle$. Compute the set Δ and the presentation $\langle B \mid \phi(w) = 1 \rangle$ for G.

Given $u, v \in A^*$, compute the normal forms $\overline{u}$ and $\overline{v}$. Then $u = v$ in M if and only if $\overline{u} = \overline{v}$ in A^*. $\square$

Divisibility problem using Zhang normal form

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For two words $u, v \in A^*$ we say u is left divisible by v if there is a word $z \in A^*$ such that $[u] = [vz]$.

Set $I = \{x \in A^+ : xy \in \Delta \text{ for some } y \in A^*\}$.

Lemma (Zhang 1992)

Let $u, v \in A^*$ be irreducible. Write $v = v'v''$ where v'' is the longest suffix of v in I^*. Then

$$\begin{aligned} u \text{ is left divisible by } v &\Leftrightarrow u \text{ is left divisible by } v' \\ &\Leftrightarrow \exists z \in A^* \text{ such that } u = v'z. \end{aligned}$$

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Proposition (Zhang 1992)

Let M be the monoid defined by $\langle A \mid w = 1 \rangle$ and let G be the group of units of M. Let $x, y \in A^*$ be irreducible words. Then $x \sim_M y$ if and only if there are words $z_1, z_2, z_3, z_4 \in A^*$ such that either

- $x = z_1 z_2, y = z_3 z_4$ with $z_2 z_1 = z_4 z_3$ in M; or
- $x = z_1 z_2, y = z_3 z_4$ such that $z_2 z_1$ and $z_4 z_3$ are invertible in M and $z_2 z_1 \sim_G z_4 z_3$ in the group of units G.

Note: It is decidable whether a word represents an invertible element of $\langle A \mid w = 1 \rangle$. An irreducible word $x \in A^*$ represents an invertible element if and only if $x \in \Delta^*$.

Geometric consequences

$M \cong \langle A \mid w = 1 \rangle$, $\quad G \cong \langle B \mid \phi(w) = 1 \rangle$ the group of units

The (right) Cayley graph $\Gamma(M, A)$ of a monoid M generated by a finite set A is the digraph with
Vertices: M Directed edges: $x \xrightarrow{a} y$ iff $y = xa$ where $x, y \in M$, $a \in A$.

The strongly connected components of $\Gamma(M, A)$ are called the Schützenberger graphs.

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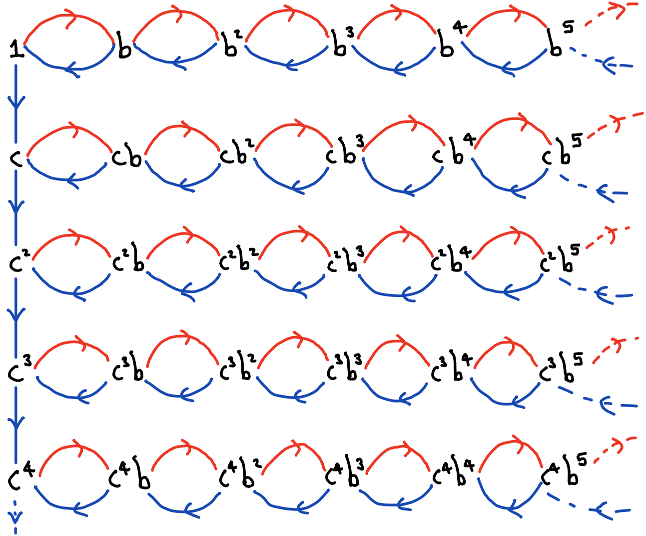
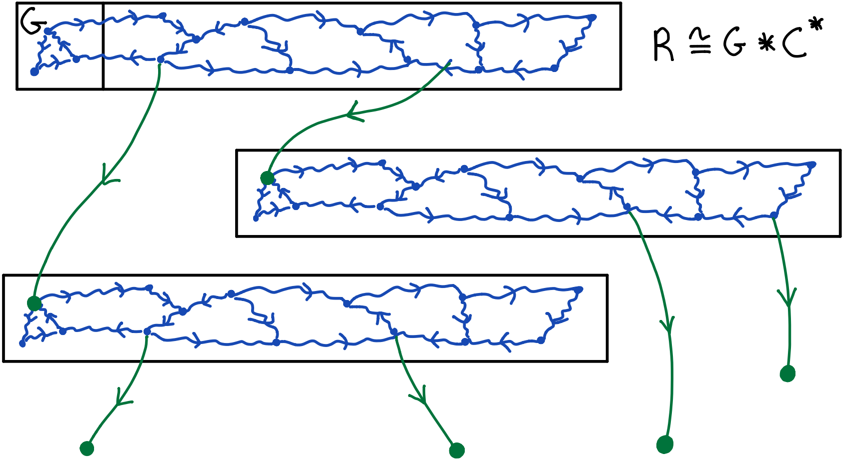
Corollary. If the group G is hyperbolic then the Cayley graph $\Gamma(M, A)$ is hyperbolic.

Bicyclic monoid $B = \langle b, c \mid bc = 1 \rangle$



$$M = \langle A \mid w = 1 \rangle$$

$$\Gamma(M, A)$$



$$R \cong G * C^*$$

Solving equations in $\langle A \mid w = 1 \rangle$

For each of the following classes is solvability of equations is decidable?

- Special one-relator monoids with torsion $\langle A \mid u^n = 1 \rangle$.
- Hyperbolic special one-relator monoids $\langle A \mid w = 1 \rangle$ (those where the group of units G is hyperbolic).
- One-relator monoids $\langle A \mid w = 1 \rangle$ where no proper prefix of w is equal to a proper suffix of w. This is the case that G is trivial e.g. the bicyclic monoid $\langle b, c \mid bc = 1 \rangle$.

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More generally we have:

Problem: Let M be the monoid defined by $\langle A \mid w = 1 \rangle$ and let G be the group of units of M. If solvability of equations is decidable in G, then does it follow that solvability of equations is decidable in M?