One-relator groups, monoids and inverse semigroups

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The word problem

Definition

A monoid M with a finite generating set A has decidable word problem if there is an algorithm which for any two words $w_1, w_2 \in A^*$ decides whether or not they represent the same element of M.

Example. $M = \mathrm{Mon}\langle a, b \mid ba = ab\rangle$ has decidable word problem.
Normal forms = $\{a^i b^j : i, j \geq 0\}$.

Some history

There are finitely presented monoids / groups with undecidable word problem.

- Markov (1947) and Post (1947), Turing (1950), Novikov (1955) and Boone (1958)

Longstanding open problem

Is the word problem decidable for one-relator monoids $\mathrm{Mon}\langle A \mid u = v\rangle$?

Word problem for one-relator groups and monoids

Groups	Monoids	Inverse monoids
$\mathrm{Gp}\langle A \mid w = 1\rangle$	$\mathrm{Mon}\langle A \mid u = v\rangle$	$\mathrm{Inv}\langle A \mid w = 1\rangle$
$\mathrm{FG}(A)/\langle\langle w \rangle\rangle$	$A^*/\langle\langle (u, v) \rangle\rangle$	$\mathrm{FIM}(A)/\langle\langle (w, 1) \rangle\rangle$

Theorem (Magnus (1932))

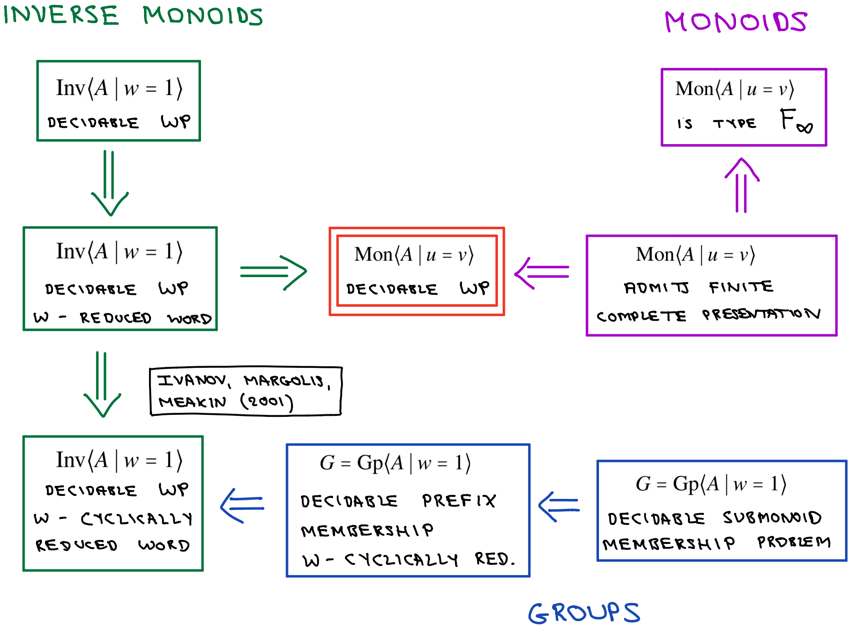
The word problem is decidable for one-relator groups.

One-relator monoids

- Word problem proved decidable in several cases by Adjan (1966), Lallament (1974), Adjan & Oganesyan (1987).

One-relator inverse monoids

- Word problem proved decidable in several cases e.g. when w satisfies...
  - Dyck word [Birget, Margolis, Meakin, 1993, 1994]
  - w-strictly positive [Ivanov, Margolis, Meakin, 2001]
  - Adjan or Baumslag-Solitar type [Margolis, Meakin, Šuniḱ, 2005]
  - Sparse word [Hermiller, Lindblad, Meakin, 2010]
  - Certain small cancellation conditions [A. Juhász, 2012, 2014]



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GROUPS

Submonoid membership problem

G - a finitely generated group with a finite group generating set A.

$\pi : (A \cup A^{-1})^* \to G$ – the canonical monoid homomorphism.

T – a finitely generated submonoid of G.

The membership problem for T within G is decidable if there is an algorithm which solves the following decision problem:

INPUT: A word $\beta \in (A \cup A^{-1})^*$.

QUESTION: $\pi(\beta) \in T$?

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There is also the uniform submonoid membership problem which takes $\beta, \alpha_1, \alpha_2, \ldots, \alpha_m \in (A \cup A^{-1})^*$ and asks $\pi(\beta) \in \mathrm{Mon}\langle \pi(\alpha_1), \ldots, \pi(\alpha_m) \rangle$?

- The submonoid membership problem is decidable in free groups $FG(A) = \mathrm{Gp}\langle A \mid \ \rangle$ by Benois (1969).
- What about for one-relator groups $\mathrm{Gp}\langle A \mid w = 1 \rangle$?

Right-angled Artin groups

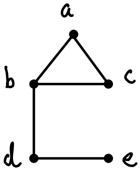
Definition

The right-angled Artin group $A(\Gamma)$ associated with the graph Γ is

$$\mathrm{Gp}\langle V\Gamma \mid uv = vu \text{ if and only if } \{u, v\} \in E\Gamma \rangle.$$

Example

$\Gamma =$



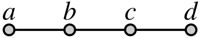
$$A(\Gamma) = \mathrm{Gp}\langle a, b, c, d, e \mid ac = ca,\ de = ed,\ ab = ba,\ bc = cb,\ bd = db \rangle$$

Right-angled Artin subgroups of one-relator groups

Theorem (RDG (2019))

$A(\Gamma)$ embeds into some one-relator group $\iff$ Γ is a finite forest.

Lohrey & Steinberg (2008) proved that $A(P_4)$ contains a finitely generated submonoid T in which membership is undecidable, where P_4 is the graph



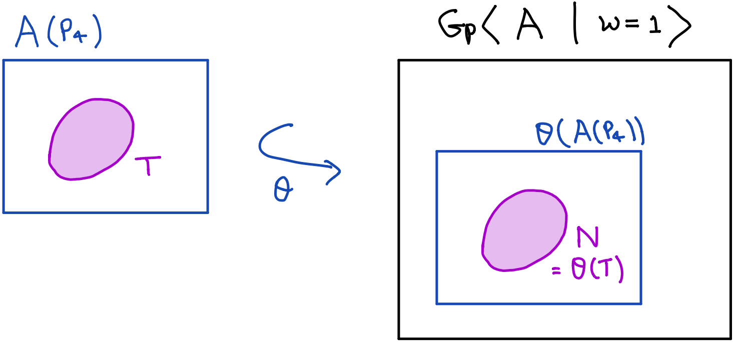
Theorem (RDG (2019))

There is a one-relator group $G = \mathrm{Gp}\langle A \mid w = 1 \rangle$ with a fixed finitely generated submonoid $N \leq G$ such that the membership problem for N within G is undecidable.

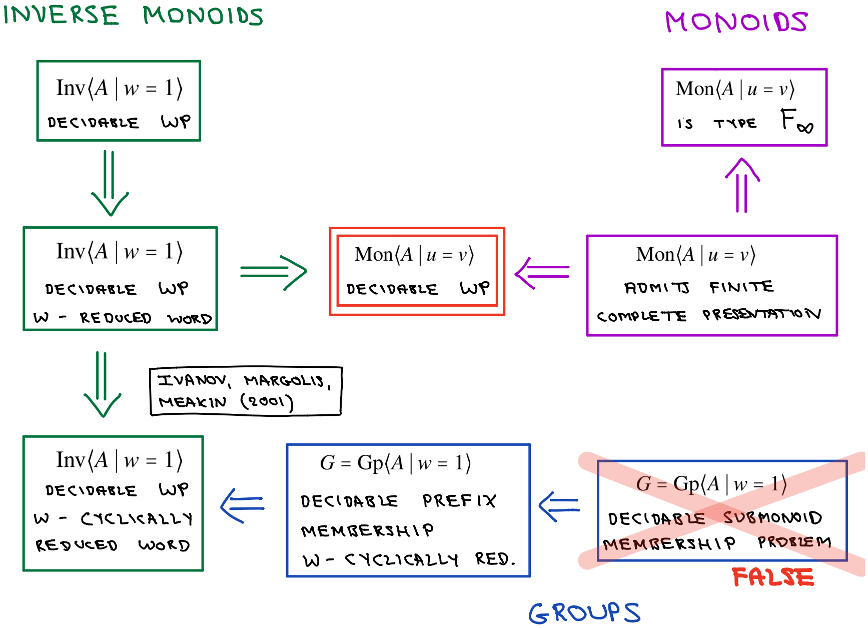
Example

$\mathrm{Gp}\langle a, t \mid atat^{-1}a^{-1}ta^{-1}t^{-1} = 1 \rangle$ is a one-relator group with this property.

Proof



- Since P_4 is a tree there is a one-relator group $G = \mathrm{Gp}\langle A \mid w = 1 \rangle$ and an embedding $\theta : A(P_4) \hookrightarrow G$.
- Then $N = \theta(T)$ is a finitely generated submonoid of G in which membership is undecidable. $\square$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GROUPS

Inverse monoid presentations

An inverse monoid is a monoid M such that for every $x \in M$ there is a unique $x^{-1} \in M$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

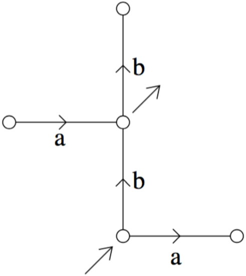
For all $x, y \in M$ we have

$$x = xx^{-1}x, \ (x^{-1})^{-1} = x, \ (xy)^{-1} = y^{-1}x^{-1}, \ xx^{-1}yy^{-1} = yy^{-1}xx^{-1} \qquad (\dagger)$$

$$\mathrm{Inv}\langle A \mid u_i = v_i \ (i \in I)\rangle = \mathrm{Mon}\langle A \cup A^{-1} \mid u_i = v_i \ (i \in I) \cup (\dagger)\rangle$$

where $u_i, v_i \in (A \cup A^{-1})^*$ and x, y range over all words from $(A \cup A^{-1})^*$.

Free inverse monoid $\mathrm{FIM}(A) = \mathrm{Inv}\langle A \mid \ \rangle$



Munn (1974)
Elements of $\mathrm{FIM}(A)$ can be represented using Munn trees. e.g. in $\mathrm{FIM}(a, b)$ we have $u = w$ where

$$u = aa^{-1}bb^{-1}ba^{-1}abb^{-1}$$
$$w = bbb^{-1}a^{-1}ab^{-1}aa^{-1}b$$

One-relator inverse monoids

A general construction

Theorem (RDG (2019))

Let $G = \mathrm{Gp}\langle B \mid u_1 = 1, \ldots, u_n = 1 \rangle$ be a finitely presented group and let N be a finitely generated submonoid of G. Then there is a finitely presented inverse monoid

$$M_{G,N} = \mathrm{Inv}\langle B, t \mid v_1 = 1, \ldots, v_n = 1 \rangle$$

with the same number of defining relations, such that

$M_{G,N}$ has decidable word problem $\iff$ The membership problem for N within G is decidable

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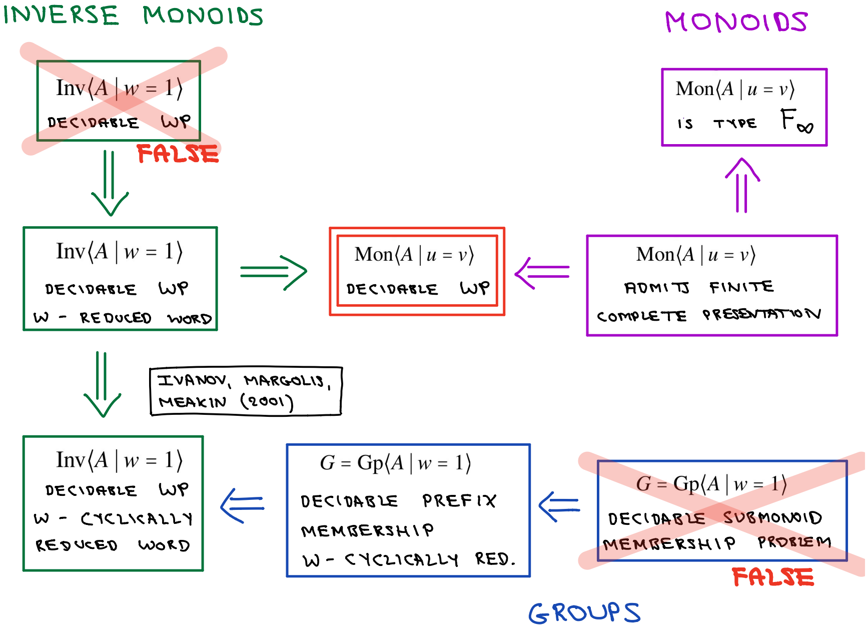
Theorem (RDG (2019))

There is a one-relator inverse monoid $\mathrm{Inv}\langle A \mid w = 1\rangle$ with undecidable word problem.

Proof: Apply the general construction above with a pair (G, N) where G is a one-relator group and $N \leq G$ is a finitely generated submonoid in which membership is undecidable. $\square$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Finite complete presentations

$$M = \mathrm{Mon}\langle A \mid u_1 = v_1,\ u_2 = v_2,\ \dots\ , u_k = v_k \rangle$$

- $w \in A^*$ is irreducible if it contains no u_i.
- The presentation is complete if there is no infinite sequence

$$w_1 \to w_2 \to w_3 \to \dots$$

with w_{i+1} obtained from w_i by applying a relation $u_r \to v_r$, and each element of the monoid M is represented by a unique irreducible word.

Example (Free commutative monoid)

$\mathrm{Mon}\langle a, b \mid ba = ab \rangle$, Normal forms (irreducibles) = $\{a^i b^j : i, j \geq 0\}$

Example (Bicyclic monoid)

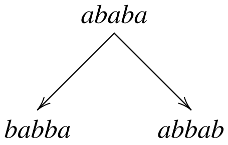
$\mathrm{Mon}\langle b, c \mid bc = 1 \rangle$, Normal forms (irreducibles) = $\{c^i b^j : i, j \geq 0\}$

Important basic fact: If a monoid M admits a finite complete presentation, then M has decidable word problem.

Example of Kupar and Narendran (1985)

- $\mathcal{P}_1 = \text{Mon}\langle a, b \mid aba = bab \rangle$

Is not a complete presentation since irreducibles not unique



However

- $\mathcal{P}_2 = \text{Mon}\langle a, b, c \mid ab = c, ca = bc, bcb = cc, ccb = acc \rangle$
- $\mathcal{P}_2$ is a complete presentation and defines the same monoid as $\mathcal{P}_1$.

Conclusion: The one-relator monoid $\text{Mon}\langle a, b \mid aba = bab \rangle$ admits a finite complete presentation.

One-relator monoids

Open problem

Does every one-relator monoid $\mathrm{Mon}\langle A \mid u = v\rangle$ admit a finite complete presentation?

- Of course, a positive answer would solve the word problem for all one-relator monoids

Anick-Groves-Squier Theorem (Anick 1986)

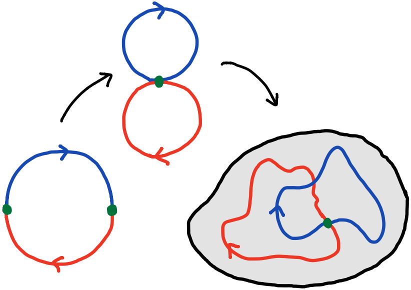
If $M = \mathrm{Mon}\langle A \mid R\rangle$ admits a finite complete presentation then M satisfies the topological finiteness property F_∞.

This motivates the following question of Kobayashi (2000)

Question: Is every one-relator monoid $\mathrm{Mon}\langle A \mid u = v\rangle$ of type F_∞?

Groups and topology

X - a space (path connected)



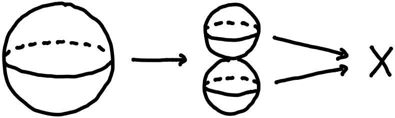
Fundamental group

$\pi_1(X)$ = { homotopy classes of loops }

Higher homotopy groups

$\pi_n(X)$ = { homotopy classes of maps $S^n \to X$ }

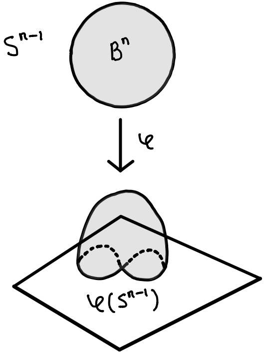
S^n the n-sphere



X is called aspherical if $\pi_n(X)$ is trivial for $n \neq 1$.

Theorem (Hurewicz (1936)) An aspherical space is determined up to homotopy equivalence by its fundamental group.

Classifying spaces of groups



CW complex - a space equipped with a sequence of subspaces

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$$

The n-skeleton X_n is obtained from X_{n-1} by attaching n-cells B^n via maps $\varphi : S^{n-1} \to X_{n-1}$.

Definition

A classifying space Y for a group G is an aspherical CW complex with fundamental group G.

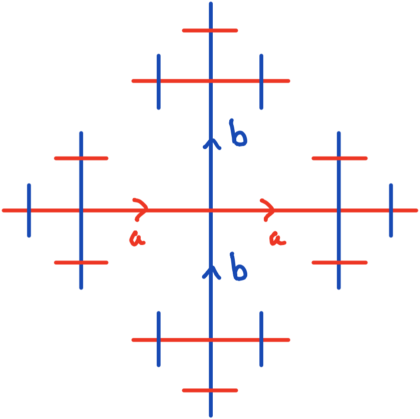
- Classifying spaces exist and are unique up to homotopy equivalence.

Whitehead theorem implies: a CW complex is aspherical $\Leftrightarrow$ its universal cover is contractible.

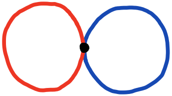
If Y is a classifying space for G then the universal cover of Y is a free G-CW complex which is contractible.

Free group

$$G = \langle a, b \mid \rangle = FG(a,b)$$



Bouquet of circles

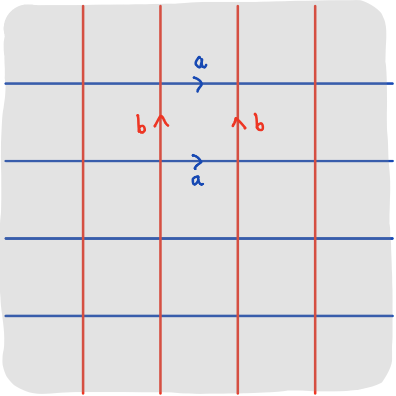


Classifying space Y for G

Universal cover X is a contractible free G-CW complex

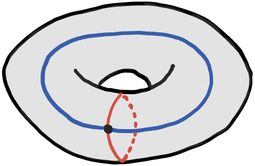
Free abelian group

$$G = \mathrm{Gp}\langle a, b \mid aba^{-1}b^{-1} = 1 \rangle \cong \mathbb{Z} \times \mathbb{Z}$$



Universal cover $X = \mathbb{R}^2$ is a contractible free G-CW complex

Torus



Classifying space Y for G

Finiteness properties

Property F_n (C. T. C. Wall (1965))

- G is of type F_n if there is a classifying space with only finitely many k-cells for each $k \leq n$.
- G is of type F_∞ if there is a classifying space with finitely many cells in all dimensions.

Examples

- G is of type $F_1 \Leftrightarrow$ it is finitely generated.
- G is of type $F_2 \Leftrightarrow$ it is finitely presented.
- $\mathbb{Z} \times \mathbb{Z}$ is of type F_∞ (finitely many cells in every dimension).

Finiteness properties of monoids

Definition (RG & Steinberg (2017))

An equivariant classifying space for a monoid M is a free M-CW complex which is contractible.

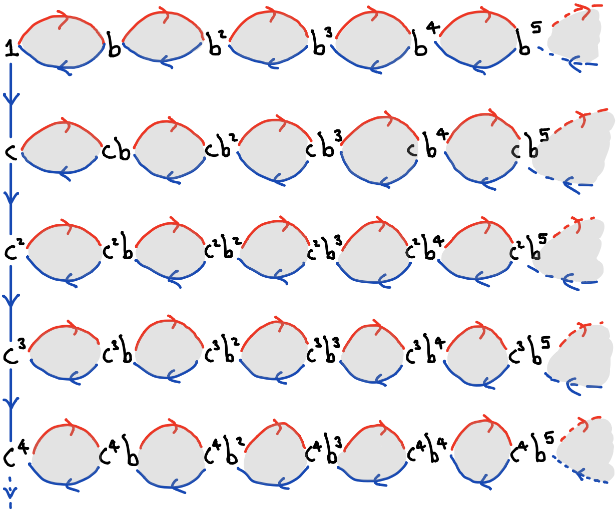
- Equivariant classifying spaces exist and are unique up to M-homotopy equivalence.

Property F_n

M is of type F_n if there is an equivariant classifying space X for M such that the set of k-cells is a finitely generated free M-set for all $k \leq n$.

- For finitely presented monoids F_n is equivalent to the homological finiteness property FP_n.

Bicyclic monoid $B = \langle b, c \mid bc = 1 \rangle$



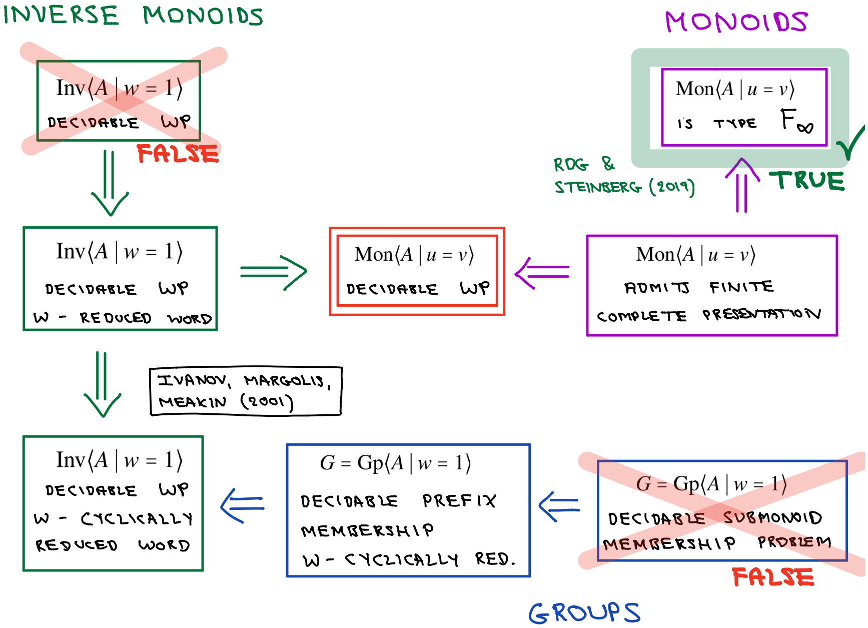
One-relator monoids

Lyndon (1950): Constructed classifying spaces for arbitrary one-relator groups, which show that every one-relator group $\mathrm{Gp}\langle A \mid w = 1 \rangle$ is of type F_∞.

Theorem (RG & Steinberg 2019)

Every one relator monoid $\mathrm{Mon}\langle A \mid u = v \rangle$ is of type F_∞.

- We prove this result by constructing equivariant classifying spaces for arbitrary one-relator monoids.
- This answers positively the question of Kobayahi (2000).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More results for one-relator inverse monoids

Key question

For which words $w \in (A \cup A^{-1})^*$ does $\mathrm{Inv}\langle A \mid w = 1 \rangle$ have decidable word problem? In particular is the word problem always decidable when w is (a) reduced or (b) cyclically reduced?

Reduced vs cyclically reduced words

$aba^{-1}ab$	- not reduced
$abba^{-1}$	- reduced but not cyclically reduced
$aba^{-1}b^{-1}$	- cyclically reduced

Definition

The prefix submonoid P_w of $\mathrm{Gp}\langle A \mid w = 1 \rangle$ is the submonoid generated by all prefixes of the word w.

Theorem (Ivanov, Margolis and Meakin (2001))

Let $w \in (A \cup A^{-1})^*$ be a cyclically reduced word. If $\mathrm{Gp}\langle A \mid w = 1 \rangle$ has decidable prefix membership problem (e.g. can decide membership in P_w) then $\mathrm{Inv}\langle A \mid w = 1 \rangle$ has decidable word problem.

Prefix membership problem via units

Theorem (Dolinka & RDG (2019))

Let $w \in (A \cup A^{-1})^*$ such that $\mathrm{Inv}\langle A \mid w = 1\rangle$ is E-unitary (e.g. true if w is cyclically reduced). Suppose that there is a finite set of words $U = \{u_1, \ldots, u_k\} \subseteq (A \cup A^{-1})^*$ such that

- each word in U represents an invertible element of $\mathrm{Inv}\langle A \mid w = 1\rangle$,
- w decomposes as $w \equiv w_1 w_2 \ldots w_n$ where each $w_i \in U \cup U^{-1}$, and
- each u_i contains a letter that does not appear in any other u_j.

Then $\mathrm{Gp}\langle A \mid w = 1\rangle$ has decidable prefix membership problem, and $\mathrm{Inv}\langle A \mid w = 1\rangle$ has decidable word problem.

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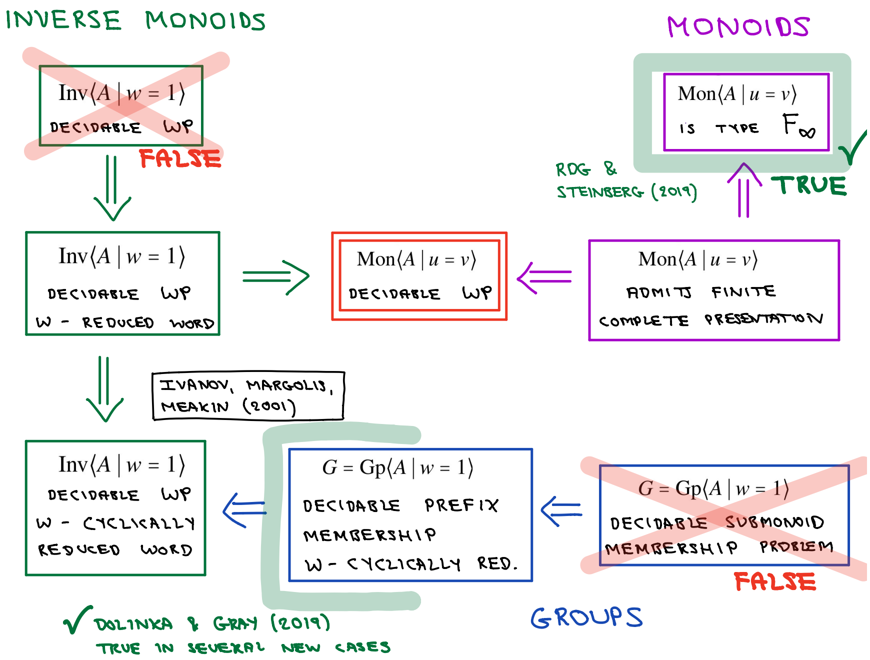
Example (Margolis, Meakin and Stephen (1987))

$$\begin{aligned} M &= \mathrm{Inv}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1\rangle \\ &= \mathrm{Inv}\langle a, b, c, d \mid (aba^{-1})(aca^{-1})(ad)(aca^{-1})(ad)(ad) \\ &\quad (aba^{-1})(aba^{-1})(aca^{-1})(ad)(aca^{-1})(ad) = 1\rangle. \end{aligned}$$

where aba^{-1}, aca^{-1} and ad are all invertible in M. Hence M has decidable word problem.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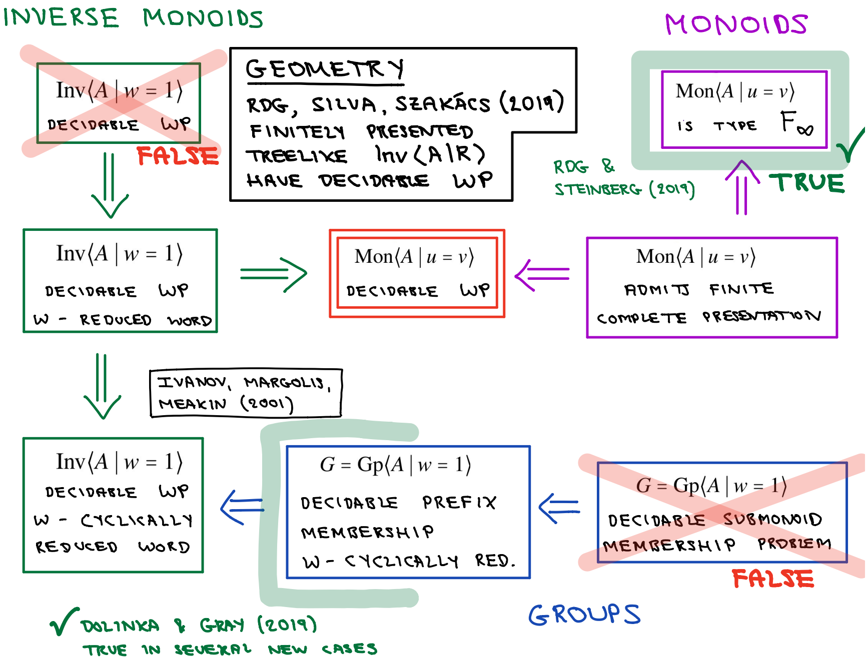
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W - REDUCED WORD
Mon⟨A | u = v⟩
DECIDABLE WP
Mon⟨A | u = v⟩
ADMITS FINITE
COMPLETE PRESENTATION
IVANOV, MARGOLIS,
MEAKIN (2001)
Inv⟨A | w = 1⟩
DECIDABLE WP
W - CYCLICALLY
REDUCED WORD
G = Gp⟨A | w = 1⟩
DECIDABLE PREFIX
MEMBERSHIP
W - CYCLICALLY RED.
G = Gp⟨A | w = 1⟩
DECIDABLE SUBMONOID
MEMBERSHIP PROBLEM
FALSE
DOLINKA & GRAY (2019)
TRUE IN SEVERAL NEW CASES