Locally finite graphs with more than one end

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Outline

Graphs

Definition

- A graph Γ is a pair $(V\Gamma, E\Gamma)$
  - $V\Gamma$ - vertex set
  - $E\Gamma$ - set of 2-element subsets of $V\Gamma$, the edge set.
- If $\{u, v\} \in E\Gamma$ we say that u and v are adjacent writing $u \sim v$.
- The neighbourhood of u is $\Gamma(u) = \{v \in V\Gamma : v \sim u\}$.
- The degree of u is $|\Gamma(u)|$.
- A graph Γ is locally finite if all of its vertices have finite degree.

Rays and ends

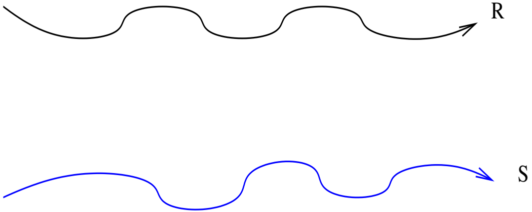
Definition

A ray in a graph Γ is a sequence $\{v_i\}_{i \in \mathbb{N}}$ of *distinct* vertices such that $v_i \sim v_{i+1}$ for all $i \in \mathbb{N}$.

The ends of a graph Γ are equivalence classes of rays.

Definition

The rays R and S are said to belong to the same end of the graph Γ if there is a third ray T such that $|R \cap T| = |S \cap T| = \aleph_0$.



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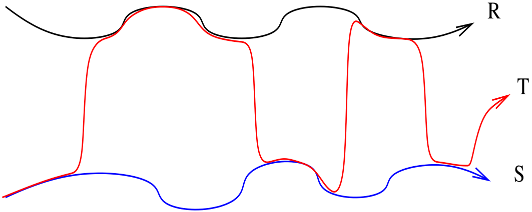
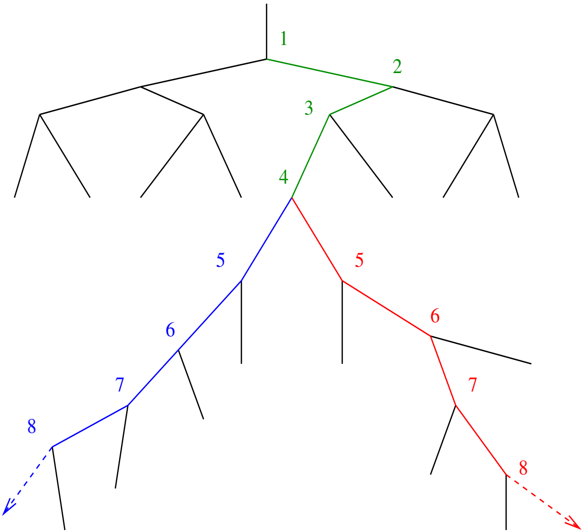


Example: 3-regular tree



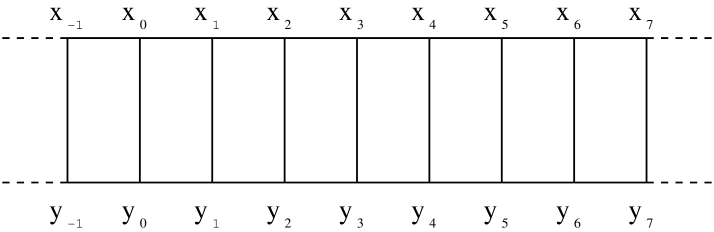
R = red ray

B = blue ray

R and B belong to different ends

Example: a 2-ended graph

Example. $\mathbb{Z}_2 \times \mathbb{Z}$ with two equivalent rays



$R = (x_0, x_1, x_2, \ldots)$ and $S = (y_0, y_1, y_2, \ldots)$ belong to the same end since with $T = (x_0, x_1, y_1, y_2, x_2, x_3, y_3, y_4, x_4, x_5, y_5, \ldots)$ we have

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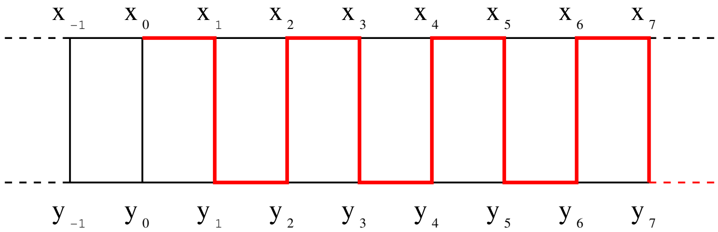

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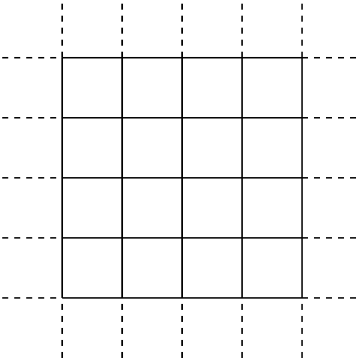
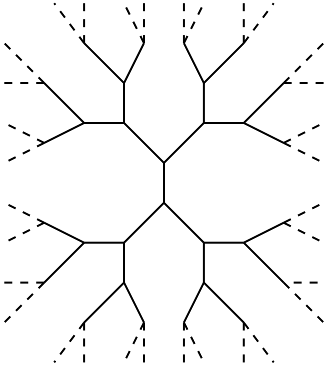
Rays and ends

- The ends describe how the graph branches.
- Each end corresponds to a “way of going to infinity”.
- If $F \subseteq \Gamma$ is finite and R is a ray then there is exactly one connected component C of $\Gamma \setminus F$ such that $|R \cap C| = \aleph_0$.
- Two rays R and S are not in the same end of Γ if and only if there is a finite set F of vertices and distinct components C and D of $\Gamma \setminus F$ such that $|R \cap C| = |S \cap D| = \aleph_0$.
- The number of ends is the least upper bound (possibly ∞) of the number of infinite connected components that can be obtained by removing finitely many vertices.

Examples: A grid, a tree and a line



Grid has 1 end



Tree has $2^{\aleph_0}$ ends

Line has 2 ends

Automorphisms

Fact

By König's infinity lemma any connected infinite locally finite graph has at least one ray and therefore at least one end.

Definition

- $G = \operatorname{Aut}\Gamma$ - the full automorphism group of Γ
- Γ is vertex transitive if G acts transitively on $V\Gamma$.

Theorem (Diestel, Jung, Möller (1993))

A connected vertex transitive graph has either 1, 2 *or* $2^{\aleph_0}$ *ends.*

Some other transitivity conditions

Distance - $d(u, v) :=$ minimum length of a path from u to v ($u, v \in V\Gamma$)

Definition

- Γ is *k-distance-transitive* if for each i with $0 \leq i \leq k$, $\mathrm{Aut}\,\Gamma$ acts transitively on the set $\{(u, v) \in V\Gamma \times V\Gamma : d(u, v) = i\}$.
- An *s-arc* is a sequence $v_0, \ldots, v_s$ of vertices such that v_i is adjacent to v_{i+1} for all $0 \leq i \leq s - 1$, and $v_j \neq v_{j+2}$ for $0 \leq j \leq s - 2$.
- A graph is *s-arc transitive* if its automorphism group acts transitively on s-arcs.

Fact

Locally finite graphs with more than one end are very sensitive to transitivity conditions such as these.

Results

Let Γ be a locally finite connected graph with more than one end.

Theorem (Möller (92))

If Γ is 2-distance transitive then Γ is k-distance transitive for all $k \in \mathbb{N}$.

Theorem (Thomassen–Woess (93))

If Γ is 2-arc transitive then Γ is a regular tree.

Theorem (Thomassen–Woess (93))

If Γ is 1-arc transitive and all vertices have degree r, where r is a prime, then Γ is a regular tree.

All these results can be proved using of the theory of structure trees.

Boundaries and tight cuts

Let $e \subseteq V\Gamma$ and let $e^* := V\Gamma \setminus e$.

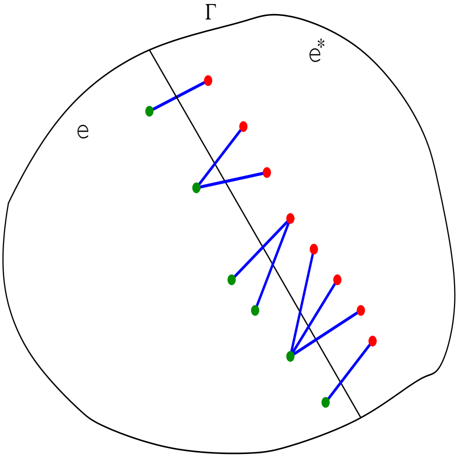
Definition

Boundary ∂e (red vertices):
$\{v \in e^* : \exists u \in e, u \sim v\}$

Co-boundary δe (blue edges):
$\{a \in E\Gamma : e \cap a \neq \varnothing \& \ e^* \cap a \neq \varnothing\}$

$e \subseteq V\Gamma$ is a cut if $|\partial e| < \aleph_0$

A cut e is tight if both e and its complement e^* are connected



Undirected tree sets

Definition

$E \subseteq \{A \subseteq V\Gamma : |\delta A| < \aleph_0\}$ is an undirected tree set if $\varnothing, V\Gamma \notin E$ and:

1. (no crossing cuts) for all $e, f \in E$ one of the following holds
$$e \subseteq f, \qquad e \subseteq f^*, \qquad e^* \subseteq f, \quad \text{or} \quad e^* \subseteq f^*$$
2. (finite intervals) $\forall e, f \in E$ there are only finitely many $g \in E : e \subset g \subset f$;
3. (complement closed) If $e \in E$ then $e^* \in E$, for all $e \in E$.

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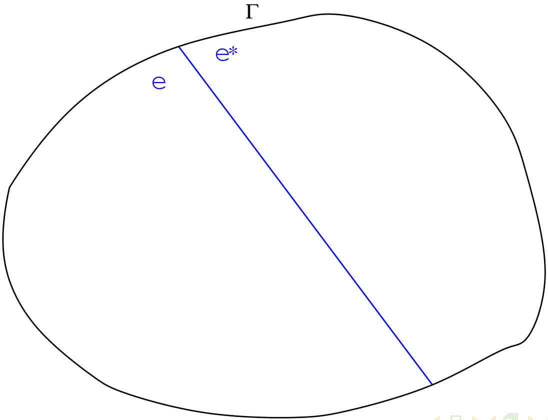

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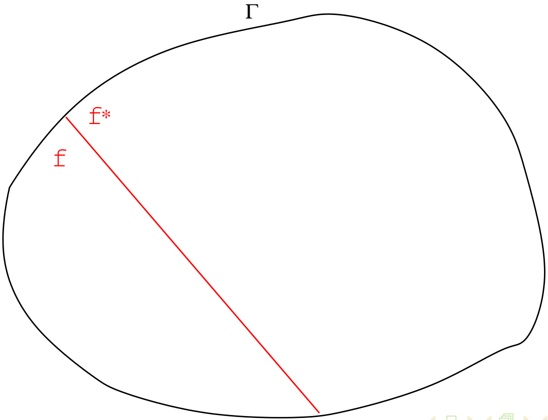

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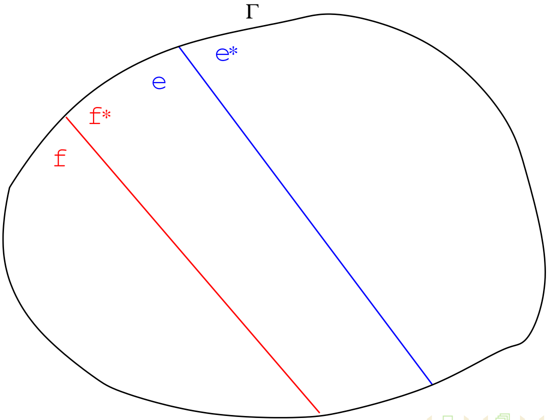

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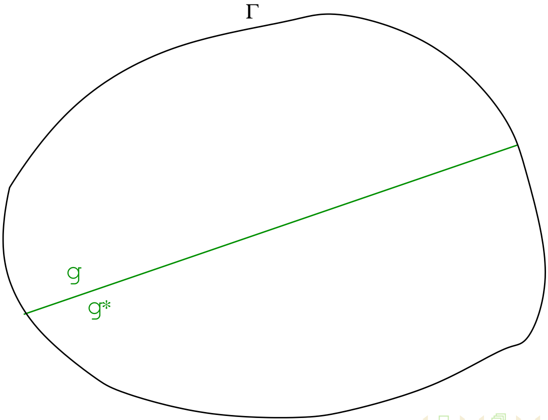

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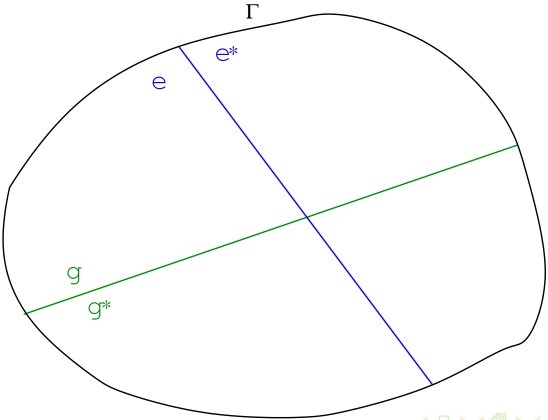


Cutting up graphs

Definition

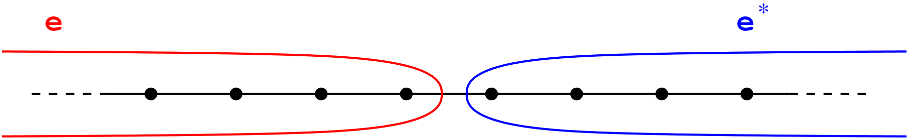
Let $G = \mathrm{Aut}\,\Gamma$. If e is an infinite tight cut, with infinite complement, such that $E = Ge \cup Ge^*$ is a tree set, then we say that e is a D-cut.

Theorem (Dunwoody (1982))

Any infinite connected graph with more than one end has a D-cut.

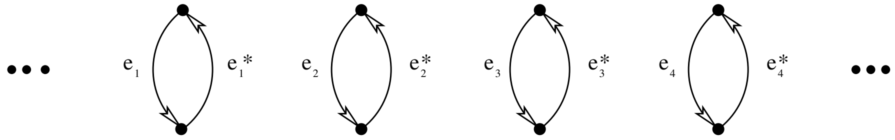
If e is a D-cut then $E = Ge \cup Ge^*$ is a tight undirected G-invariant tree set.

Example. A D-cut e in the graph $\Gamma = \mathbb{Z}$



Structure tree

- We use a tight undirected G-invariant tree set E to construct a tree
- $T(E)$ - has directed edges that come in pairs with edges in one-one correspondence with the sets in the tree set E
- $e = (u, v)$ in $T(E) \Rightarrow (v, u)$ in $T(E)$ and is labelled by the complement e^* of e
- $Y :=$ the disjoint union of pairs of edges $\{e, e^*\}$ for $e \in E$



Structure tree

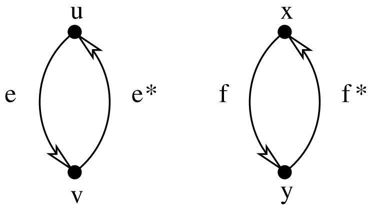
Identify various vertices

$f \ll e$ if $f \subset e \;\&\; \neg[\exists g \in E : f \subset g \subset e]$

Equivalence relation $\approx$ on VY
$e = (u, v), f = (x, y) \in Y$ write $v \approx x$ if
$x = v$ or if $f \ll e$

Pairs of edges are never identified

Define $T = T(E) := Y/\approx$.



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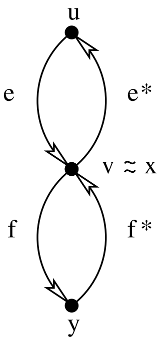

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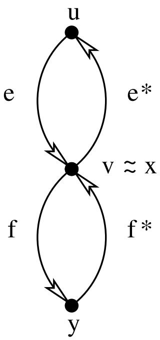
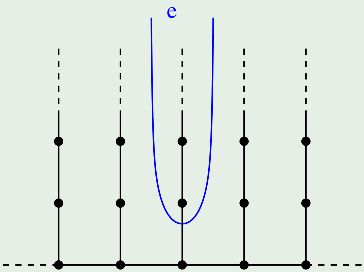
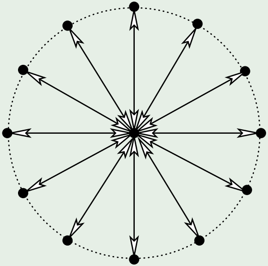


- T is connected and has no cycles of length greater than 2.
- If $e = e_0, e_1, \ldots, e_n = f$ is a directed edge path in T then in Γ we have $e_0 \supseteq e_1 \supseteq \ldots \supseteq e_n$.

Example: An infinite comb and a tree set



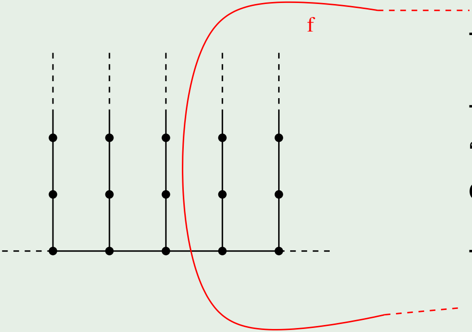
The comb Γ



Structure tree $T(E)$
where $E = Ge \cup Ge^*$

- E consists of the “teeth” of the comb and their complements
- The edges radiating out from the centre of $T(E)$ correspond to the “teeth” of the comb.

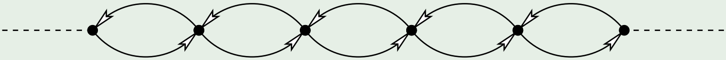
Example: An infinite comb and a different tree set



The tree set is $F := Gf \cup Gf^*$

The tree set F consists of "translates" of f and their complements.

The corresponding structure tree $T(F)$ is a line, drawn below.



$T(F)$

The structure mapping

Definition ($\phi : V\Gamma \to T(E)$)

For $v \in V\Gamma$ let $e(v) \subseteq E$ be the set of members of E to which v belongs.

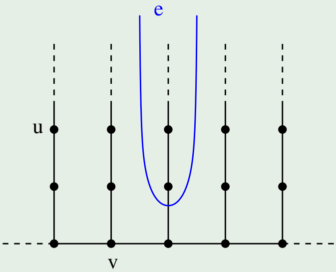
Let $T(v) \subseteq T(E)$ be the subgraph of T induced by the directed edges $e(v)$.

For each edge pair $f = (x, y)$, $f^* = (y, x)$ either $f \in e(v)$ or $f^* \in e(v)$.

Fact. There is a unique vertex w in $T(v)$ whose out degree is zero. Every edge in $T(v)$ "points towards" the vertex w.

Define $\phi(v) := w$.

Example: Structure map 1



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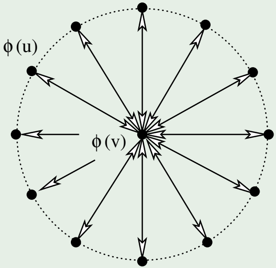
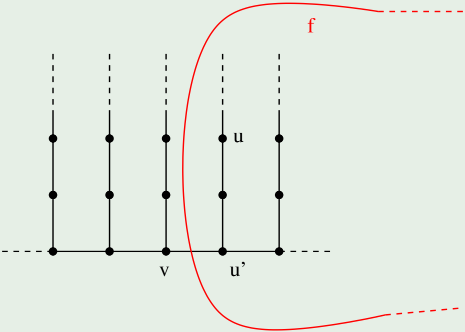


Structure tree $T(E)$ where $E = Ge \cup Ge^*$

- u belongs to its tooth and to the complements of all other teeth.
- v belongs to the complement of every tooth.

Example: Structure map 2



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u and u' both belong to f, all of its translates to the left (supersets), but none of its translates to the right (subsets).

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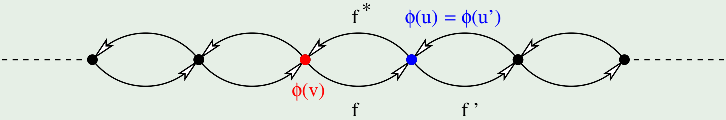

Properties of $\phi : V\Gamma \to T(E)$

Proposition

The mapping ϕ has the following properties:

- If $u, v \in \Gamma$, and $e \in E$ with $u \in e$ and $v \in e^*$, then it follows that $\phi(u) \neq \phi(v)$.
- If $\phi(u) = \phi(v)$ then we cannot distinguish between u and v just using the tree set E.
- $G = \operatorname{Aut}\Gamma$ acts on the tree $T(E)$.
- The action of G commutes with the map ϕ.

Applying the ideas to prove a result

Theorem (Thomassen–Woess (93))

Let Γ be a connected locally finite 1-arc transitive graph all of whose vertices have degree r, where r is a prime. If Γ has more than one end then Γ is a regular tree.

Proof outline

- By Dunwoody's theorem Γ has a D-cut $e_0 \subseteq V\Gamma$
- Set $E := Ge_0 \cup Ge_0^*$, $T = T(E)$, and fix $v \in V\Gamma$

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- **Claim 1.** $\Gamma(v)$ has more than one σ-class.
- **Claim 2.** Any two σ-classes have the same size.
- **Corollary.** $|\Gamma(v)|$ is prime $\Rightarrow$ the σ-classes are all trivial.

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  $\exists c \in \mathbb{N} \setminus \{0\} : \{x, y\} \in E\Gamma \Rightarrow d_T(\phi(x), \phi(y)) = c$

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Applying the ideas to prove a result

Theorem (Thomassen–Woess (93))

Let Γ be a connected locally finite 1-arc transitive graph all of whose vertices have degree r, where r is a prime. If Γ has more than one end then Γ is a regular tree.

Proof outline

- Γ 1-arc-transitive $\Rightarrow$
  $\exists c \in \mathbb{N} \setminus \{0\} : \{x, y\} \in E\Gamma \Rightarrow d_T(\phi(x), \phi(y)) = c$
- **Claim 3.** Γ does not have any cycles
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- $\therefore$ $k > 1 \rightarrow v_0 \not\sim v_k$.

My own work, cycle-free partial orders

- The other theorems above can be proved using structure trees.

Definition

A graph Γ is *k-CS-transitive* if for any finite isomorphic connected substructures A and B, of size k, there is an automorphism $\alpha \in \operatorname{Aut} \Gamma$ such that $A^\alpha = B$ (setwise).

- We have been considering k-CS-transitive bipartite graphs arising from, so called, cycle-free partial orders.
- Such bipartite graphs are "tree-like", and for locally finite graphs cycle-freeness implies > 1 end.
- Using D-cuts and structure trees we classified the connected 3-CS-transitive locally finite graphs with more than one end.