Crystal monoids

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Plactic monoid via Knuth relations

Definition

Let $\mathcal{A}_n$ be the finite ordered alphabet $\{1 < 2 < \ldots < n\}$.
Let $\mathcal{R}$ be the set of defining relations:

$$\begin{aligned} zxy &= xzy \quad \text{and} \quad yzx = yxz \qquad && x < y < z, \\ xyx &= xxy \quad \text{and} \quad xyy = yxy \qquad && x < y. \end{aligned}$$

The Plactic monoid $\mathrm{Pl}(A_n)$ is the monoid defined by the presentation $\langle \mathcal{A}_n | \mathcal{R} \rangle$.

That is, $\mathrm{Pl}(A_n) = \mathcal{A}_n^* / \sim$ where $\sim$ is the smallest congruence on the free monoid $\mathcal{A}_n^*$ containing $\mathcal{R}$.

- We call $\sim$ the Plactic congruence. The relations in this presentation are called the Knuth relations.

The Plactic monoid

- Has origins in work of Schensted (1961) and Knuth (1970) concerned with combinatorial problems on Young tableaux.
- Later studied in depth by Lascoux and Shützenberger (1981).

Due to close relations to Young tableaux, has become a tool in several aspects of representation theory and algebraic combinatorics.

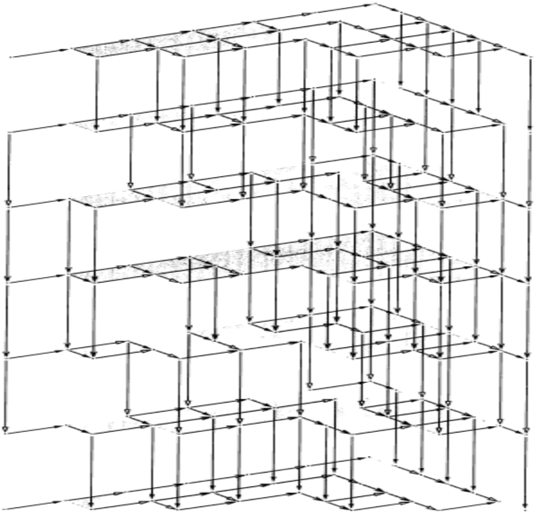
$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 4 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 4 & 5 & 5 \\ \cline{1-3} 6 & 8 \\ \cline{1-2} \end{array} \qquad \longleftrightarrow \qquad \mathrm{w}(T) = 4213512581246$$

Fact: The set of word readings of tableaux is a set of normal forms for the elements of the Plactic monoid. So $\mathrm{Pl}(A_n)$ is the monoid of tableaux:

Elements The set of all tableaux over $\mathcal{A}_n = \{1 < 2 < \cdots < n\}$.

Products Computed using Schensted insertion algorithm.

Crystals



[1]

[1]Fig 8.4 from Hong and Kang's book *An introduction to quantum groups and crystal bases.*

Crystal graphs

(following Kashiwara and Nakashima (1994))

Idea: Define a directed labelled digraph Γ_{A_n} with the properties:

- Vertex set = $\mathcal{A}_n^*$
- Each directed edge is labelled by a symbol from the label set $I = \{1, 2, \ldots, n-1\}$.
- For each vertex $u \in \mathcal{A}_n^*$ every $i \in I$ there is at most one directed edge labelled by i leaving u, and there is at most one directed edge labelled by i entering u,

$$u \xrightarrow{\quad i \quad} v \,, \qquad\qquad w \xrightarrow{\quad i \quad} u$$

- If $u \xrightarrow{\quad i \quad} v$ then $|u| = |v|$, so words in the same component have the same length as each other. In particular, connected components are all finite.

Building the crystal graph Γ_{A_n}

$$\mathcal{A}_n = \{1 < 2 < \ldots < n\}$$

We begin by specifying structure on the words of length one

$$1 \xrightarrow{\quad 1 \quad} 2 \xrightarrow{\quad 2 \quad} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n$$

This is known as a Crystal basis.

Kashiwara operators on letters

For each $i \in \{1, \ldots, n-1\}$ we define partial maps $\tilde{e}_i$ and $\tilde{f}_i$ on the letters $\mathcal{A}_n$ called the Kashiwara crystal graph operators. For each edge

$$a \xrightarrow{\quad i \quad} b \quad ,$$

we define $\tilde{f}_i(a) = b$ and $\tilde{e}_i(b) = a$.

Kashiwara operators on words

Let $u \in \mathcal{A}_n^*$ and $i \in I$.

- Are $\tilde{e}_i(u)$ or $\tilde{f}_i(u)$ defined? If so what words do we obtain?

Example with $\mathcal{A}_3 = \{1 < 2 < 3\}$

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3$$

$$a \xrightarrow{i} \tilde{f}_i(a) \quad , \quad \tilde{e}_i(b) \xrightarrow{i} b$$

Let $u = 33212313232$ and let $i = 2 \in I = \{1, 2\}$.

$$3 \quad 3 \quad 2 \quad 1 \quad 2 \quad 3 \quad 1 \quad 3 \quad 2 \quad 3 \quad 2$$

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$$\begin{array}{ccccccccccc} 3 & 3 & 2 & 1 & 2 & 3 & 1 & 3 & 2 & 3 & 2 \\ & & + & & + & & & & + & & + \end{array}$$

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$$\begin{matrix} 3 & 3 & 2 & 1 & 2 & 3 & 1 & 3 & 2 & 3 & 2 \\ - & - & + & & + & - & & - & + & - & + \end{matrix}$$

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3	3	2	1	2	3	1	3	2	3	2
$-$	$-$	$+$		$+$	$-$		$-$	$+$	$-$	$+$
$-$	$-$	~~$+$~~		~~$+$~~	~~$-$~~		~~$-$~~	~~$+$~~	~~$-$~~	$+$

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−	−	+		+	−		−	+	−	+
−	−	~~+~~		~~+~~	~~−~~		~~−~~	~~+~~	~~−~~	+
−	−									+

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3 3 2 1 2 3 1 3 2 3 3 $= \tilde{f}_2(u)$

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$-$	$-$									$+$

3 3 2 1 2 3 1 3 2 3 3 $= \tilde{f}_2(u)$

3 2 2 1 2 3 1 3 2 3 2 $= \tilde{e}_2(u)$

The crystal graph Γ_{A_n}

Definition

The crystal graph Γ_{A_n} is the directed labelled graph with:

- Vertex set: $\mathcal{A}_n^*$
- Directed labelled edges: for $u \in \mathcal{A}_n^*$

$$u \xrightarrow{\ i\ } \tilde{f}_i(u) \quad , \qquad \tilde{e}_i(u) \xrightarrow{\ i\ } u$$

Note: When defined $\tilde{e}_i(\tilde{f}_i(u)) = u$ and $\tilde{f}_i(\tilde{e}_i(u)) = u$.

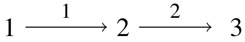
Crystal graph components for $\mathcal{A}_3 = \{1 < 2 < 3\}$

Word length one

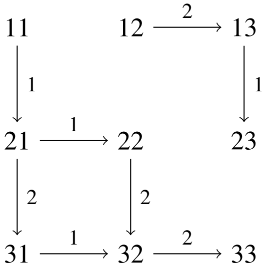
$$1 \xrightarrow{\;1\;} 2 \xrightarrow{\;2\;} 3$$

Crystal graph components for $\mathcal{A}_3 = \{1 < 2 < 3\}$

Word length one

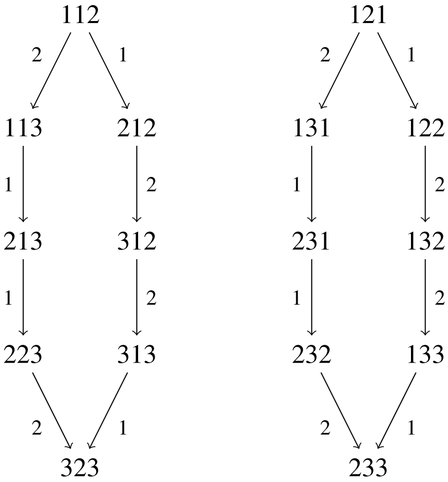


Word length two



Crystal graph components for $\mathcal{A}_3 = \{1 < 2 < 3\}$

Word length three



Plactic monoid via crystals

Definition: Two connected components $B(w)$ and $B(w')$ of Γ_{A_n} are isomorphic if there is a label-preserving digraph isomorphism $f : B(w) \to B(w')$.

Fact: In Γ_{A_n} if $B(w) \cong B(w')$ then there is a unique isomorphism $f : B(w) \to B(w')$.

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Theorem (Kashiwara and Nakashima (1994))

Let Γ_{A_n} be the crystal graph with crystal basis

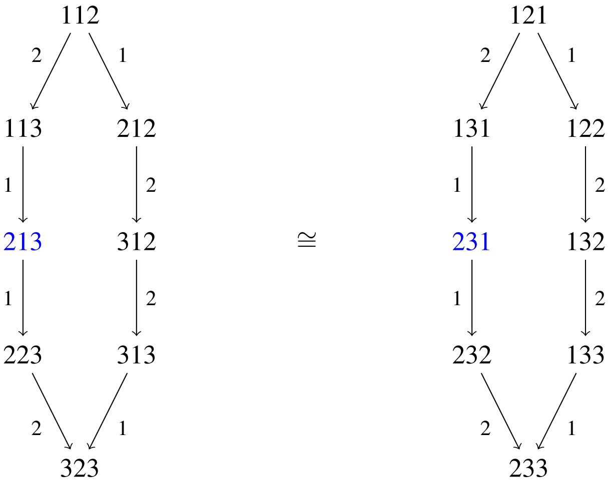
$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n$$

Define a relation $\sim$ on $\mathcal{A}_n^*$ by

$$u \sim w \Leftrightarrow \exists \text{ an isomorphism } f : B(u) \to B(w) \text{ with } f(u) = w.$$

Then $\sim$ is the Plactic congruence and $\mathrm{Pl}(A_n) = \mathcal{A}_n^* / \sim$ is the Plactic monoid.

Crystal graph components for $\mathcal{A}_3 = \{1 < 2 < 3\}$

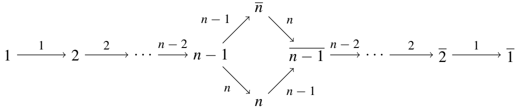


Where do crystals come from?

J. Hong, S.-J. Kang,
Introduction to Quantum Groups and Crystal Bases.
Stud. Math., vol. 42, Amer. Math. Soc., Providence, RI, 2002.

- Take a “nice” Lie algebra $\mathfrak{g}$ e.g. a finite-dimensional semisimple Lie algebra.
- Crystal bases are bases of $U_q(\mathfrak{g})$-modules satisfying certain axioms.
  - $U_q(\mathfrak{g})$ = quantum deformation of universal enveloping algebra $U(\mathfrak{g})$ (Drinfeld and Jimbo (1985)).
- Every crystal basis has the structure of a coloured digraph (called a crystal graph). The structure of these coloured digraphs has been explicitly determined for certain semisimple Lie algebras (special linear, special orthogonal, symplectic, some exceptional types).
- Crystal constructed using Kashiwara operators is a combinatorial tool for studying representations of $U_q(\mathfrak{g})$.

Crystal bases and crystal monoids

Lie algebra type	Crystal basis	Monoid
A_n: $\mathfrak{sl}_{n+1}$	$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n$	$\mathrm{Pl}(A_n)$
B_n: $\mathfrak{so}_{2n+1}$	$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-1} n \xrightarrow{n} 0 \xrightarrow{n} \overline{n} \xrightarrow{n-1} \cdots \xrightarrow{2} \overline{2} \xrightarrow{1} \overline{1}$	$\mathrm{Pl}(B_n)$
C_n: $\mathfrak{sp}_{2n}$	$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-1} n \xrightarrow{n} \overline{n} \xrightarrow{n-1} \cdots \xrightarrow{2} \overline{2} \xrightarrow{1} \overline{1}$	$\mathrm{Pl}(C_n)$
D_n: $\mathfrak{so}_{2n}$		$\mathrm{Pl}(D_n)$
G_2	$1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{1} 0 \xrightarrow{1} \overline{3} \xrightarrow{2} \overline{2} \xrightarrow{1} \overline{1}$	$\mathrm{Pl}(G_2)$

Known results and our interest

Known results on crystals A_n, B_n, C_n, D_n, or G_2 and their crystal monoids:

1. Crystal bases - combinatorial description Kashiwara and Nakashima (1994).
2. Tableaux theory and Schensted-type insertion algorithms - Kashiwara and Nakashima (1994), Lecouvey (2002, 2003, 2007).
3. Finite presentations for $\mathrm{Pl}(X)$ via Knuth-type relations - Lecouvey (2002, 2003, 2007).

Theory we have been developing for these monoids:

4. Finite complete rewriting systems
   - Finite presentation with ordered relations $u \to_R v$ where each word converges $w \to_R^* \overline{w}$ to unique normal form.
5. Automatic structures
   - Regular language of normal forms such that $\forall a \in A\ \exists$ a finite automaton recognising pairs of normal forms that differ by multiplication by a.

Our results

A. J. Cain, R. D. Gray, A. Malheiro
Crystal bases, finite complete rewriting systems, and biautomatic structures for Plactic monoids of types A_n, B_n, C_n, D_n, and G_2.
arXiv:math.GR/1412.7040, 50 pages.

Theorem

For any $X \in \{A_n, B_n, C_n, D_n, G_2\}$, there is a finite complete rewriting system (Σ, T) that presents $\mathrm{Pl}(X)$.

Theorem

The monoids $\mathrm{Pl}(A_n)$, $\mathrm{Pl}(B_n)$, $\mathrm{Pl}(C_n)$, $\mathrm{Pl}(D_n)$, and $\mathrm{Pl}(G_2)$ are all biautomatic.

Corollary

The monoids $\mathrm{Pl}(A_n)$, $\mathrm{Pl}(B_n)$, $\mathrm{Pl}(C_n)$, $\mathrm{Pl}(D_n)$, and $\mathrm{Pl}(G_2)$ all have word problem solvable in quadratic time.

Current and future work

- Further develop the theory of crystal monoids in general
  - We can obtain other examples (e.g. bicyclic monoid is a crystal monoid).
  - They all have decidable word problem.
  - Under what conditions do they admit finite complete rewriting systems / are automatic?
- What do our results say about the Plactic algebras of Littelmann?

P. Littelmann,
A Plactic Algebra for Semisimple Lie Algebras.
Advances in Mathematics 124 (1996), 312–331.

- Investigate how our results might be applied to give new computational tools for working with crystals (e.g. using rewriting systems / finite automata to compute with crystals).