On regularity and the word problem for free idempotent generated semigroups

Robert Gray
(joint work with I. Dolinka and N. Ruskuc)

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University of East Anglia

Observations

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this is what makes people interesting.

And that is the reason we fall in love...

The word problem for semigroups and groups

Definition

A semigroup S with a finite generating set A has decidable word problem if there is an algorithm which for any two words $w_1, w_2 \in A^+$ decides whether or not they represent the same element of S.

Example. $S \cong \langle a, b \mid ab = ba \rangle$ has decidable word problem.

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Some history

- Markov (1947) and Post (1947): first examples of finitely presented semigroups with undecidable word problem;
- Turing (1950): finitely presented cancellative semigroup with undecidable word problem;
- Novikov (1955) and Boone (1958): finitely presented group with undecidable word problem.

The word problem for $\mathsf{IG}(E)$

S - semigroup, $E = E(S)$

Free idempotent generated semigroup:

$$\mathsf{IG}(E) = \langle E \mid e \cdot f = ef \text{ where } \underbrace{ef = e \text{ or } ef = f \text{ or } fe = e \text{ or } fe = f}_{\text{Basic pairs } (e,f)} \rangle$$

Relates to theory of biordered sets of idempotents (Nambooripad (1979)).

If E is finite then this is a finite presentation defining $\mathsf{IG}(E)$.

Question

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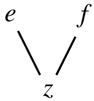
Let us consider two illustrative examples:

1. Three-element semilattice
2. Four-element rectangular band

Example 1: Three-element meet semilattice

$S:$



Multiplication: $x \cdot y = \inf(x, y)$

$e^2 = e,\ f^2 = f,\ z^2 = z$

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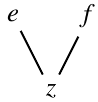

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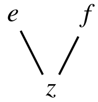

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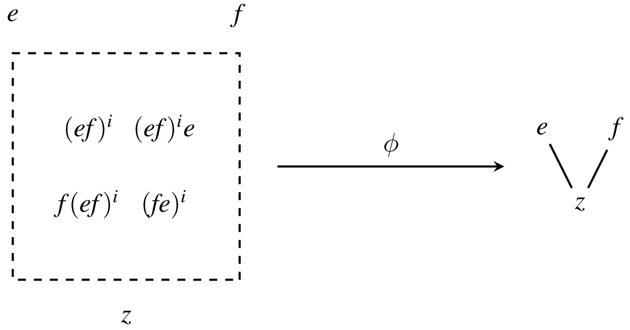
$\mathsf{IG}(E)$ $\qquad\qquad$ $S = E(S) = E$



Example 2: Four-element rectangular band

$$S: \quad \begin{array}{ccc} e_{11} & — & e_{12} \\ | & & | \\ e_{21} & — & e_{22} \end{array}$$

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$e_{ij}e_{kl} = e_{il}$ (note $e_{ij}^2 = e_{ij}$)

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$$\mathsf{IG}(E) = \langle e_{ij}\ (i,j \in \{1,2\}) \mid e_{ij}e_{il} = e_{il} \quad \text{(same row)}$$
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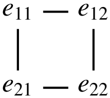

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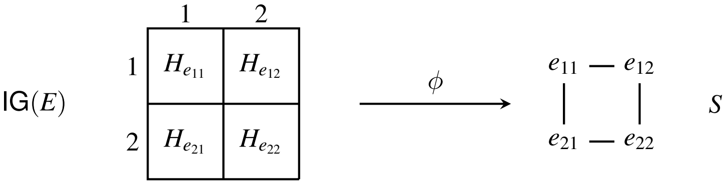


- $\mathsf{IG}(E) \cong \{1,2\} \times \mathbb{Z} \times \{1,2\}$ where $\mathbb{Z} = \langle a, a^{-1}\rangle$ with multiplication

$$(i, a^m, j)(k, a^n, l) = (i, a^m p_{jk} a^n, l), \quad P = \begin{pmatrix} 1 & 1 \\ 1 & a \end{pmatrix}.$$

This is a Rees matrix semigroup over $\mathbb{Z}$ with structure matrix P.

Behaviour exhibited in these examples

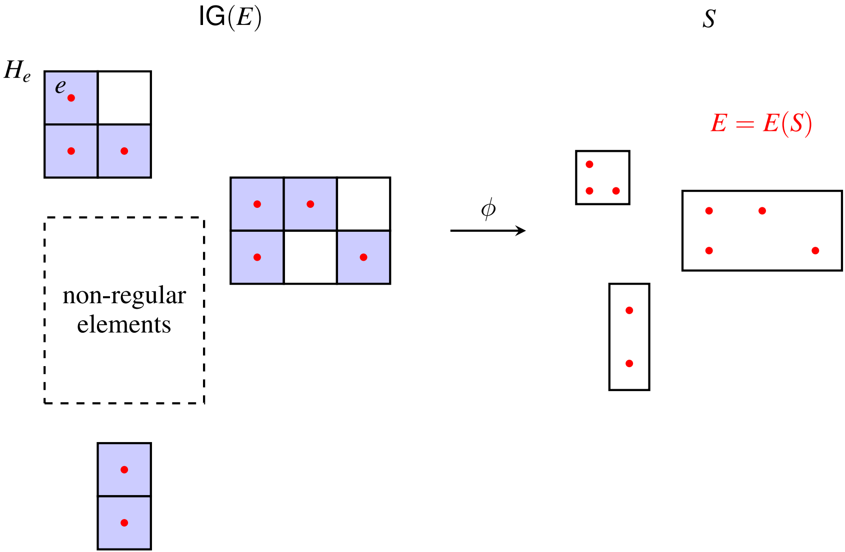
Semilattice example

- The regular part of $\mathsf{IG}(E)$ is finite and the "same as" the original finite semigroup S.
- Analysis of the non-regular part of $\mathsf{IG}(E)$ is necessary to solve the word problem. The word problem is decidable because the non-regular part is well behaved.

Rectangular band example

- There are no non-regular elements in $\mathsf{IG}(E)$.
- Because the subgroups of $\mathsf{IG}(E)$ that arise are well behaved (they are all isomorphic to $\mathbb{Z}$ and thus have decidable word problem) it follows that $\mathsf{IG}(E)$ has decidable word problem.

The general picture



Embedding a group with undecidable word problem

S - semigroup, $E = E(S)$

Question

Does $\mathsf{IG}(E)$ have decidable word problem if E is finite?

General facts

- E finite $\Rightarrow$ every maximal subgroup of $\mathsf{IG}(E)$ is finitely presented.
- If $\mathsf{IG}(E)$ has decidable word problem then every maximal subgroup of $\mathsf{IG}(E)$ must have decidable word problem.

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Theorem (RG & Ruskuc (2012))

Every finitely presented group is a maximal subgroup of some free idempotent generated semigroup arising from a finite semigroup.

Since there exist finitely presented groups with undecidable word problem...

Corollary

There is a finite semigroup S such that $\mathsf{IG}(E)$ has undecidable word problem.

Word problem for regular words

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New question

Does $\mathsf{IG}(E)$ have decidable word problem if E is finite and every maximal subgroup of $\mathsf{IG}(E)$ has decidable word problem?

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Theorem (Dolinka, RG, Ruskuc (2014))

If E is finite and every maximal subgroup of $\mathsf{IG}(E)$ has decidable word problem then there is an algorithm which, given any two words $u, v \in E^+$

1. decides whether both u and v represent regular elements of $\mathsf{IG}(E)$ and, if they do,
2. decides whether $u = v$ in $\mathsf{IG}(E)$.

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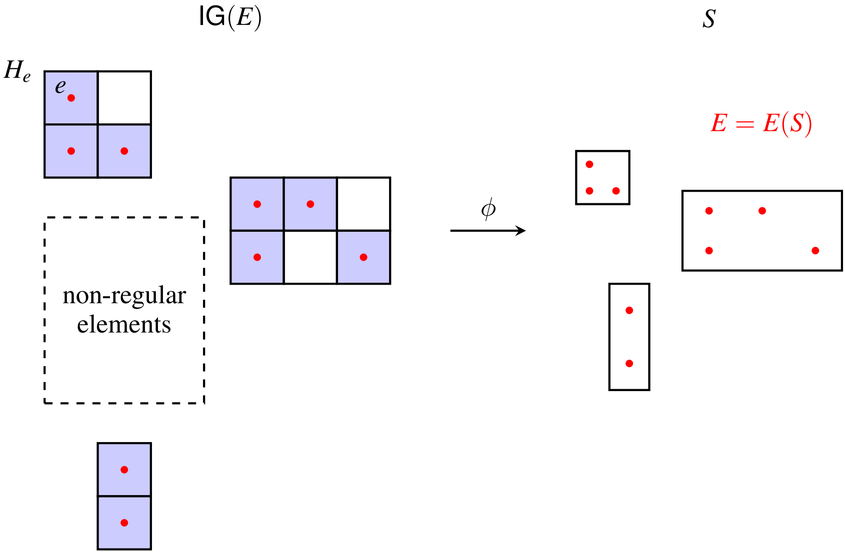

Word problem in general

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Does $\mathsf{IG}(E)$ have decidable word problem if E is finite and every maximal subgroup of $\mathsf{IG}(E)$ has decidable word problem?

Theorem (Dolinka, RG, Ruskuc (2014))

There exists a finite band $B_{G,H}$ such that:

(i) All maximal subgroups of $\mathsf{IG}(B_{G,H})$ have decidable word problem.

(ii) The word problem for $\mathsf{IG}(B_{G,H})$ is undecidable.

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For this result we make use of another decision problem...

The membership problem for subgroups

Definition

Let G be a group with finite generating set A, and let H be a subgroup of G given by a finite set of words which generate H.

Then the membership problem for H in G is the problem of deciding, for an arbitrary word w over the generators A, whether or not w represents an element of the subgroup H.

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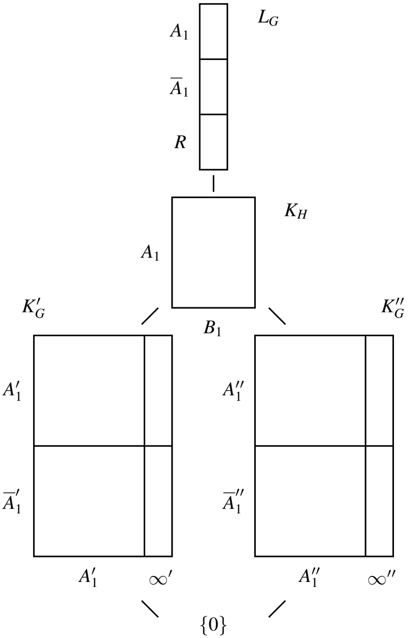
Theorem (Mihailova (1958))

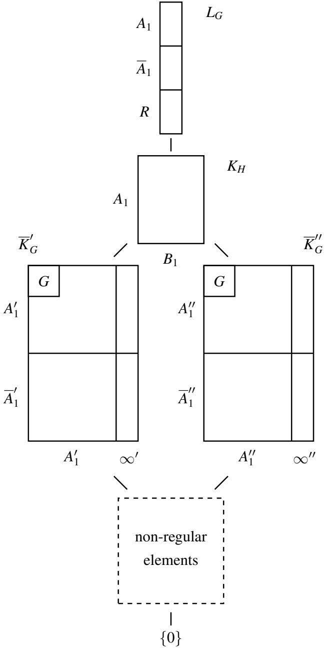
Their exists a finitely presented group G with a finitely generated subgroup H such that

- G has decidable word problem, but
- the membership problem for H in G is undecidable.

The $B_{G,H}$ construction



Encoding the membership problem



Structure of $\mathsf{IG}(B_{G,H})$

Each of $\overline{K}'_G$ and $\overline{K}''_G$ is a Rees matrix semigroup over G

$$\overline{K}'_G \cong I' \times G \times J', \quad \overline{K}''_G \cong I'' \times G \times J''.$$

For any word w over A the equality $(1', 1, 1')(1'', 1, 1'') = (1', w^{-1}, 1')(1'', w, 1'')$ holds in $\mathsf{IG}(B_{G,H}) \Leftrightarrow w \in H$.

Conclusion: If $\mathsf{IG}(B_{G,H})$ had decidable word problem this would imply the membership problem for H in G is decidable, which is a contradiction ↯

Further questions

S - semigroup, $E = E(S)$ with E finite

Free idempotent generated semigroup:

$$\mathsf{IG}(E) = \langle E \mid e \cdot f = ef \ \ (e,f \in E,\ \{e,f\} \cap \{ef,fe\} \neq \emptyset)\ \rangle$$

We have seen that:

- If all maximal subgroups of $\mathsf{IG}(E)$ have decidable word problem then
  - The word problem for regular elements in $\mathsf{IG}(E)$ is decidable.
  - The word problem for $\mathsf{IG}(E)$ is not decidable in general.

Problem

Find necessary and sufficient conditions for $\mathsf{IG}(E)$ to have decidable word problem.

For this it may be useful to investigate the Schützenberger groups of $\mathsf{IG}(E)$.