The geometry of Schützenberger graphs and subgroups of special inverse monoids

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(joint work with Mark Kambites)

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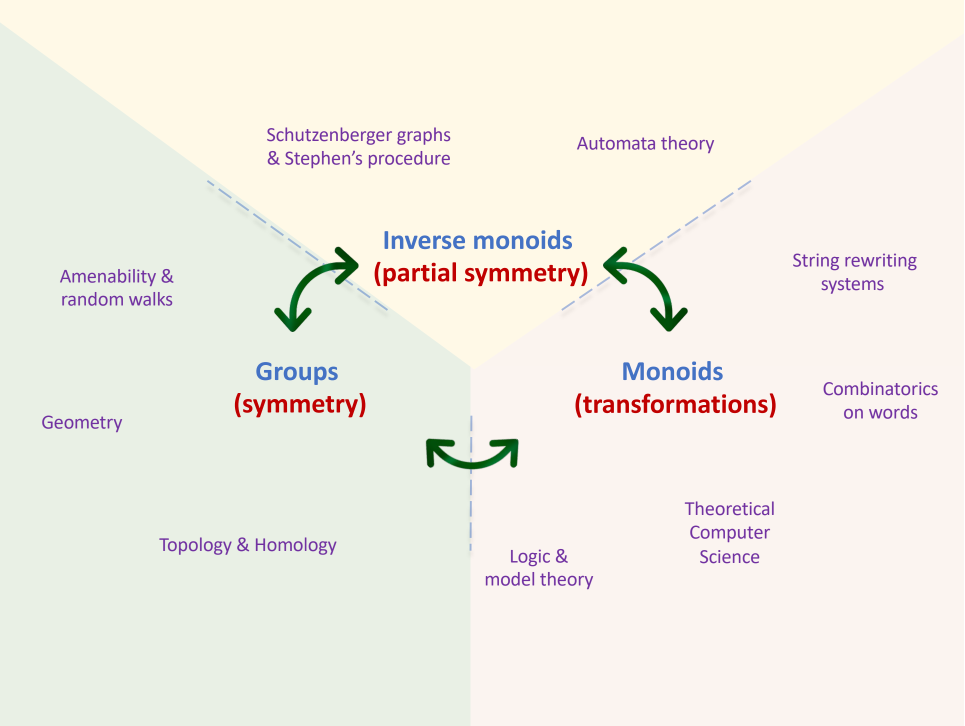

Schutzenberger graphs
& Stephen's procedure
Automata theory
Inverse monoids
(partial symmetry)
Amenability &
random walks
String rewriting
systems
Groups
(symmetry)
Monoids
(transformations)
Geometry
Combinatorics
on words
Theoretical
Computer
Science
Topology & Homology
Logic &
model theory

Presentations

$$\langle A \mid R \rangle = \langle \underbrace{a_1, \ \ldots, \ a_n}_{\text{letters / generators}} \mid \underbrace{u_1 = v_1, \ \ldots, \ u_m = v_m}_{\text{words / defining relations}} \rangle$$

- Defines the monoid $M = A^*/\rho$ where ρ is the equivalence relation on the free monoid A^* of all words over A where two words are in the same equivalence class (i.e. they represent the same element of M) if one can be transformed into the other by applying the relations R.

Example

The free group F_2 of rank 2 is defined by

$$\langle a, a^{-1}, b, b^{-1} \mid aa^{-1} = 1, \ a^{-1}a = 1, \ bb^{-1} = 1, \ b^{-1}b = 1 \rangle$$

In F_2 we have

$$abb^{-1}ab = aab = aaaa^{-1}b = aaab^{-1}ba^{-1}b$$

but $ab \neq aab$.

Presentations

$$\mathrm{Mon}\langle A \mid R\rangle = \mathrm{Mon}\langle \underbrace{a_1, \ldots, a_n}_{\text{letters / generators}} \mid \underbrace{u_1 = v_1, \ldots, u_m = v_m}_{\text{words / defining relations}} \rangle$$

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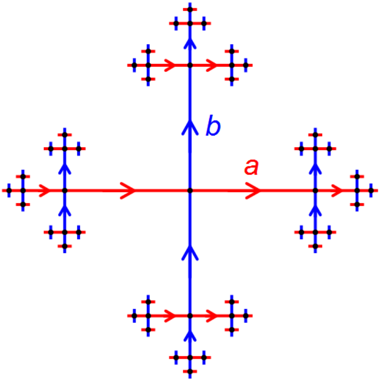
Elements of the free group F_2

$$F_2 = \mathrm{Mon}\langle a, a^{-1}, b, b^{-1} \mid aa^{-1} = 1,\ a^{-1}a = 1,\ bb^{-1} = 1,\ b^{-1}b = 1\rangle$$



Cayley graph of F_2

Vertices: Elements of F_2.
Edges: $g \xrightarrow{x} gx$ for $g \in F_2$ and $x \in A$.

Fact: Every word w over the generators is equal in F_2 to a unique reduced word $\mathrm{red}(w)$ with no occurrences of xx^{-1} or $x^{-1}x$.

e.g. $abb^{-1}aab^{-1}ba^{-1}b = aab$.

This solves the word problem in F_2:

$w_1 = w_2$ in $F_2 \Leftrightarrow \mathrm{red}(w_1) = \mathrm{red}(w_2)$ as words

The reduced words give normal forms for elements of the free group.

Group presentations

$$G \cong \mathrm{Gp}\langle A \mid u_i = v_i \ (i \in I)\rangle \quad (u_i, v_i \in (A \cup A^{-1})^*)$$

Elements of G - equivalence classes of words over $A \cup A^{-1}$ where

$u = v$ in G $\Leftrightarrow$ we can transform u into v by applying defining relations or relations from the free group.

Example

The free group F_2 is defined by $\mathrm{Gp}\langle a, b \mid \ \rangle$.

Example

The free abelian group $\mathbb{Z} \times \mathbb{Z}$ is defined by $\mathrm{Gp}\langle a, b \mid ab = ba\rangle$.
Normal forms: $a^i b^j \ (i, j \in \mathbb{Z})$

Definition

G is finitely presented if $G \cong \mathrm{Gp}\langle A \mid R\rangle$ with $|A| < \infty$ and $|R| < \infty$.

Subgroups of finitely presented groups

Question: What are the finitely generated subgroups of finitely presented groups?

- There are finitely generated subgroups of finitely presented groups that are not finitely presented
  - e.g. Grunewald (1978): shows $F_2 \times F_2$ has such a subgroup.
- Neumann (1937): There are uncountably many 2-generator groups. $\Rightarrow$ Not every finitely generated group embeds in a finitely presented group.

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Computing subsets of $(A \cup A^{-1})^*$

- A set W of words is computably enumerable if there is an algorithm which takes any word u as input and, if u is a member of W, then the algorithm eventually halts and says YES; otherwise it runs forever.
- A set W of words is computable if there is an algorithm which takes any word u as input, terminates after a finite amount of time and decides whether or not the word u belongs to W, returning either YES or NO.

Important fact: There exist sets of words that are c.e. but not computable.

The Higman Embedding Theorem

Definition

A recursive presentation for a finitely generated group is a presentation on a finite number of generators such that the set of defining relators is computably enumerable.

Theorem (Higman (1961))

A finitely generated group G can be embedded in some finitely presented group if and only if G can be recursively presented.

Example

For any computably enumerable subset S of $\mathbb{N}$

$$\mathrm{Gp}\langle a, b, c, d \mid a^{-i} b a^{i} = c^{-i} d c^{i} \ (i \in S) \rangle$$

is recursively presented and hence embeds in a finitely presented group.

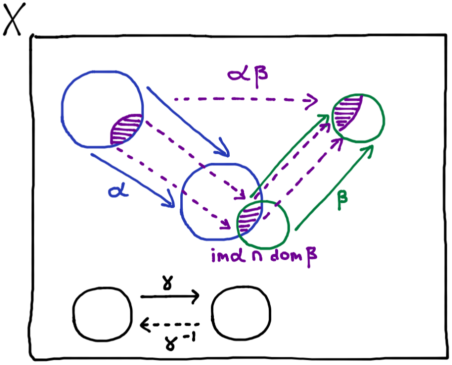
- Choosing S to be computably enumerable but non-computable
  $\Rightarrow$ This group has undecidable word problem.
  $\Rightarrow$ There are finitely presented groups with undecidable word problem.

Inverse monoids

An inverse monoid is a monoid M such that for every $x \in M$ there is a unique $x^{-1} \in M$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

$m \in M$ is a right unit if there is an $n \in M$ such that $mn = 1$, left unit is defined analogously, and a unit is an element that is both a left and right unit.

Example: I_X = monoid of all partial bijections $X \to X$



Examples: In I_3

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & - & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ - & 1 & - \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ - & 1 & 2 \end{pmatrix}$$

Note:
$\gamma\gamma^{-1} = \mathrm{id}_{\mathrm{dom}\gamma}$

Group of units of I_X: is the symmetric group S_X.

Inverse monoid presentations

An inverse monoid is a monoid M such that for every $x \in M$ there is a unique $x^{-1} \in M$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

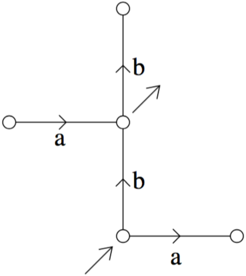
For all $x, y \in M$ we have

$$x = xx^{-1}x, \ (x^{-1})^{-1} = x, \ (xy)^{-1} = y^{-1}x^{-1}, \ xx^{-1}yy^{-1} = yy^{-1}xx^{-1} \qquad (\dagger)$$

$$\mathrm{Inv}\langle A \mid u_i = v_i \ (i \in I)\rangle = \mathrm{Mon}\langle A \cup A^{-1} \mid u_i = v_i \ (i \in I) \cup (\dagger)\rangle$$

where $u_i, v_i \in (A \cup A^{-1})^*$ and x, y range over all words from $(A \cup A^{-1})^*$.

Free inverse monoid $\mathrm{FIM}(A) = \mathrm{Inv}\langle A \mid \rangle$



Munn (1974)
Elements of $\mathrm{FIM}(A)$ can be represented using Munn trees. e.g. in $\mathrm{FIM}(a, b)$ we have $u = w$ where

$u = aa^{-1}bb^{-1}ba^{-1}abb^{-1}$
$w = bbb^{-1}a^{-1}ab^{-1}aa^{-1}b$

Special inverse monoids

Definition

A finitely presented special inverse monoid is one defined by a presentation of the form

$$\mathrm{Inv}\langle A \mid w_1 = 1, \dots, w_k = 1 \rangle.$$

Motivation from the theory of one-relator monoids and groups

Theorem (Ivanov, Margolis, Meakin (2001))

If the word problem is decidable for all inverse monoids of the form $\mathrm{Inv}\langle A \mid r = 1 \rangle$, with r a reduced word, then the word problem is also decidable for every one-relation monoid $\mathrm{Mon}\langle A \mid u = v \rangle$.

Theorem (Adjan (1966))

The group of units G of a one-relator monoid $M = \mathrm{Mon}\langle A \mid r = 1 \rangle$ is a one-relator group. Furthermore, M has decidable word problem.

Aim: Study the subgroup structure of finitely presented special inverse monoids.

Units of special inverse monoids

Monoids

Theorem (Makanin (1966))

The group of units G of $M = \mathrm{Mon}\langle A \mid r_1 = 1, \ldots, r_k = 1\rangle$ admits a k-relator presentation. Furthermore, M has decidable word problem if and only if G has decidable word problem.

Example

The group of units of $M = \mathrm{Mon}\langle a, b, c, d \mid abab = 1, abcdabcdabcd = 1\rangle$ is $G = \mathrm{Gp}\langle X, Y \mid X^2 = 1, (XY)^3 = 1\rangle$.

Inverse monoids

Theorem (Ivanov, Margolis, Meakin (2001))

The group of units G of $M = \mathrm{Inv}\langle A \mid r_1 = 1, \ldots, r_k = 1\rangle$ is finitely generated.

Theorem (RDG & Ruškuc (2021))

There is a finitely presented special inverse monoid $\mathrm{Inv}\langle A \mid r_1 = 1, \ldots, r_k = 1\rangle$ whose group of units is not finitely presented.

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Units of special inverse monoids

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Theorem (RDG & Kambites (2022))

The groups of units of finitely presented special inverse monoids are exactly the finitely generated, recursively presented groups.

Notes:

- Equivalently, by Higman, these are exactly the finitely generated subgroups of finitely presented groups.
- One direction of the proof is straightforward: Since the group of units is a finitely generated subgroup of a finitely presented inverse monoid it follows quickly it must itself be recursively presented.
- The other direction requires a construction to realise each such group as the group of units.
- For this we use the theory of Schützenberger graphs.

Schützenberger graphs

Definition

The Schützenberger graph $S\Gamma(1)$ of $M = \mathrm{Inv}\langle A \mid r_1 = 1, \ldots, r_k = 1 \rangle$ is the subgraph of the Cayley graph of M induced on the set of right units of M.

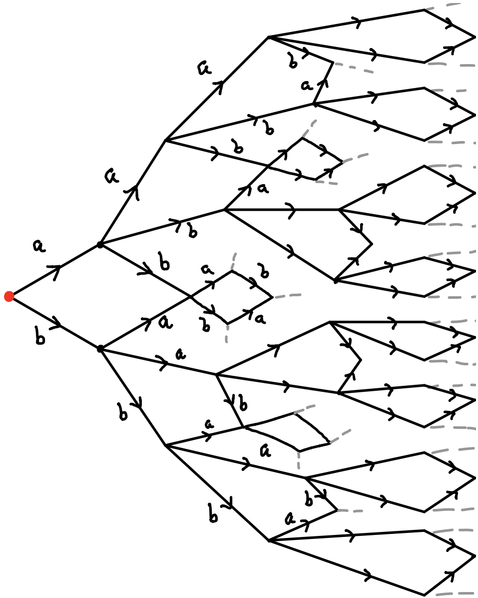
Theorem (Stephen (1990))

The group of units of $M = \mathrm{Inv}\langle A \mid r_1 = 1, \ldots, r_k = 1 \rangle$ is isomorphic to the group $\mathrm{Aut}(S\Gamma(1))$ of label-preserving automorphisms of the Schützenberger graph $S\Gamma(1)$.

Stephen's procedure

The Schützenberger graph $S\Gamma(1)$ can be obtained as the limit of a sequence of labelled digraphs obtained by an iterative construction given by successively applying operations called expansions and Stallings foldings.

Example - Stephen's Procedure



$\mathrm{Inv}\langle a, b \mid aba^{-1}b^{-1} = 1\rangle$

Stephen's procedure

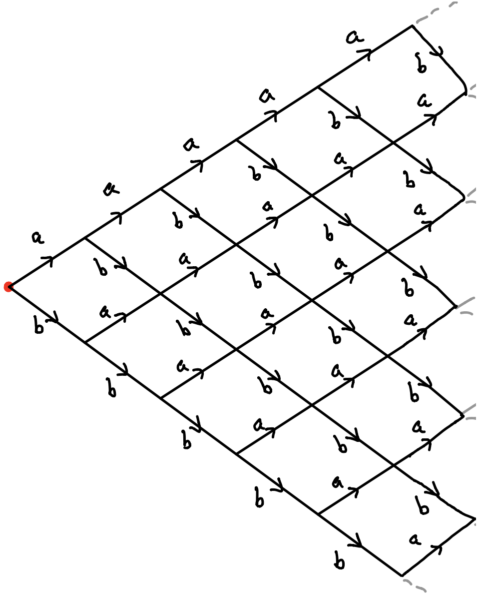
Expansions: Attach a simple closed path labelled by r at a vertex (if one does not already exist).

Stallings foldings: Identify two edges with the same label and the same initial or terminal vertex.

This process may not stop. Stephen shows that the

- process is confluent &
- limits in an appropriate sense to $S\Gamma(1)$.

Example - Stephen's Procedure



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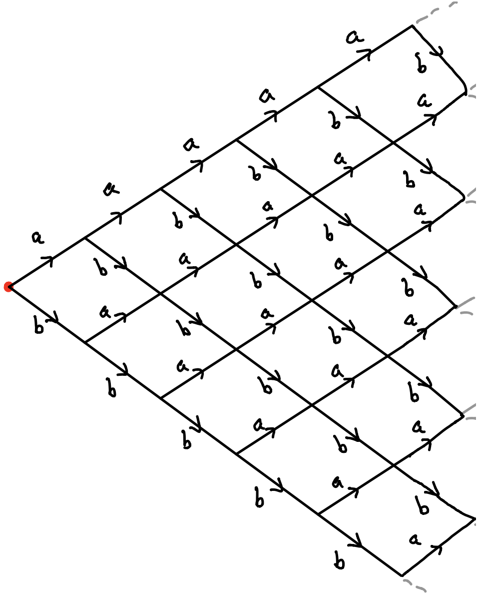

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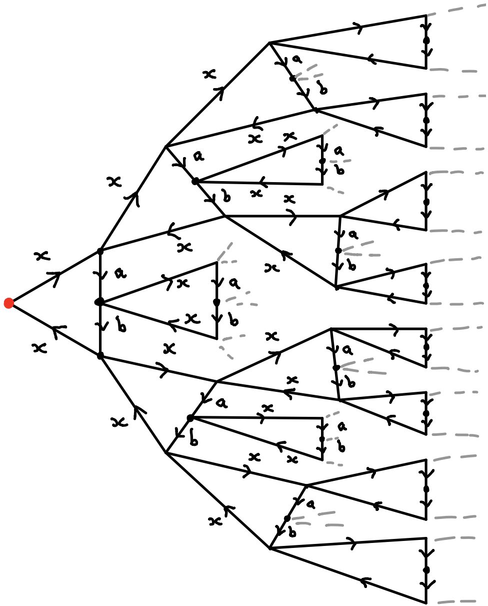
The group of units is $\mathrm{Aut}(S\Gamma(1)) = \{1\}$.

Example - a non-trivial group of units



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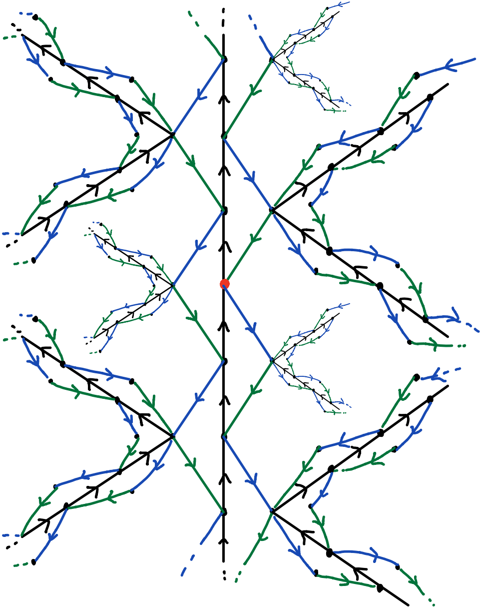


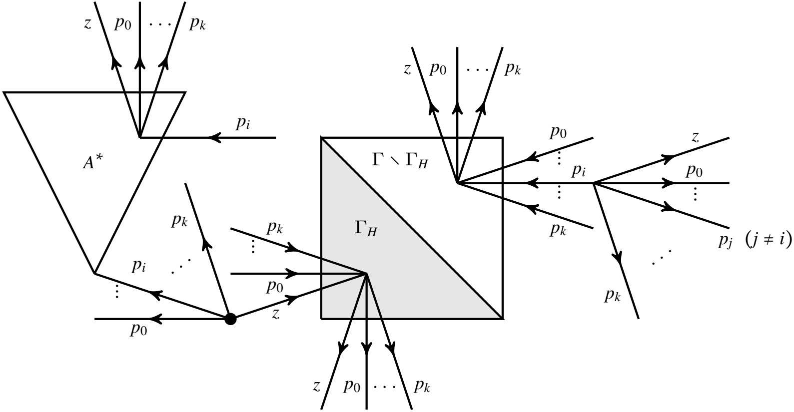
$\mathrm{Inv}\langle a, b, x \mid xabx = 1\rangle$

The group of units is

$$\mathrm{Aut}(S\Gamma(1)) \cong \mathbb{Z}$$

the infinite cyclic group.

Our construction - the general idea



- Given $G = \mathrm{Gp}\langle A \mid R \rangle$ finitely presented, and $H \leq G$ a finitely generated subgroup, we construct $\mathrm{Inv}\langle A \mid w_1 = 1, \ldots, w_k = 1 \rangle$ whose $S\Gamma(1)$ has the above structure where Γ is the Cayley graph of G with respect to A, and Γ_H is the subgraph induced on the subset H of vertices.
- We prove every automorphism of $S\Gamma(1)$ fixes Γ_H setwise and deduce $\mathrm{Aut}(S\Gamma(1)) \cong \mathrm{Aut}(\Gamma_H) \cong H$.

One-relator case

Theorem (Adjan (1966))

The group of units G of a one-relator monoid $M = \mathrm{Mon}\langle A \mid r = 1 \rangle$ is a one-relator group.

Theorem (RDG & Ruškuc (2021))

There exists a one-relator special inverse monoid $M = \mathrm{Inv}\langle A \mid r = 1 \rangle$ whose group of units G is not a one-relator group.

Question: Is the group of units of $\mathrm{Inv}\langle A \mid r = 1 \rangle$ always finitely presented?

Definition. A finitely presented group G is said to be coherent if every finitely generated subgroup of G is finitely presented.

Open problem (Baumslag (1973))

Is every one-relator group coherent?

- Louder and Wilton (2020) & independently Wise (2020) proved that one-relator groups with torsion are coherent.

Theorem (RDG & Ruškuc (2021))

If all one-relator special inverse monoids $\mathrm{Inv}\langle A \mid r = 1 \rangle$ have finitely presented groups of units then all one-relator groups are coherent.

Maximal subgroups in general

Definition

For any idempotent $e = e^2$ in an inverse monoid M define

$$H_e = \{m \in M : mm^{-1} = e = m^{-1}m\}.$$

Then H_e is a group called a group $\mathcal{H}$-class of M.
e.g. H_1 is the group of units of M.

Definition

A recursive presentation for a (countable but not necessarily finitely generated) group is a presentation of the form $\mathrm{Gp}\langle A \mid R\rangle$ where A is either finite or $A = \{a_i : i \in \mathbb{N}\}$ and R is a computably enumerable subset of $(A \cup A^{-1})^*$.

Theorem (RDG & Kambites (2022))

The possible group $\mathcal{H}$-classes of finitely presented special inverse monoids are exactly the (not necessarily finitely generated) recursively presented groups.

One-relator case (maximal subgroups)

Theorem (RDG & Kambites (2022))

Every finitely generated subgroup of a one-relator group arises as a group $\mathcal{H}$-class in a one-relator special inverse monoid.

Question: Is every group $\mathcal{H}$-class of $\mathrm{Inv}\langle A \mid r = 1 \rangle$ necessarily finitely generated?

The above question would be answered negatively if the answer to the following is yes:

Question: Does there exist a one-relator group $G = \mathrm{Gp}\langle A \mid w = 1 \rangle$ with a finitely generated subgroup $H \leq G$ and an element $g \in G$ such that $H \cap gHg^{-1}$ is not finitely generated.

- This relates to the Howson property - which asks that $H \cap K$ is finitely generated whenever H and K both are.
  - There are one-relator groups (even hyperbolic ones) that do not have the Howson property – Karrass & Solitar (1969), Kapovich (1999).