

Approaching cosets using Green's relations and Schützenberger groups

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British Mathematical Colloquium
March 2008

General question

How are the properties of a semigroup related to those of its substructures?

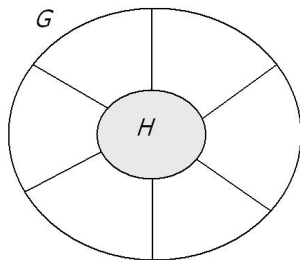
Index in group theory

G - group, H - subgroup of G

- ▶ The **right cosets** of H in G are the sets $\{Hg : g \in G\}$.
- ▶ The **index** $[G : H]$ is the number of (right) cosets of H in G .

G is the disjoint union of the cosets of H

Subgroups of **finite index** share many properties with their parent groups.



Finiteness conditions

Proposition

For groups the following finiteness conditions are preserved when taking finite index subgroups and when taking finite index extensions:

- ▶ finitely generated
- ▶ finitely presented
- ▶ locally finite
- ▶ periodic
- ▶ FP_n
- ▶ soluble word problem
- ▶ residually finite
- ▶ automatic
- ▶ finite derivation type.

Idea

- ▶ Develop a theory of index for semigroups
- ▶ Use it to gain a better understanding of the relationship between the properties of a semigroup and those of its substructures.

Subgroups of monoids: translational index

S - monoid, K - subgroup of S

- ▶ The **right cosets** of K are the elements of the strong orbit of K under the action of S by right multiplication. So

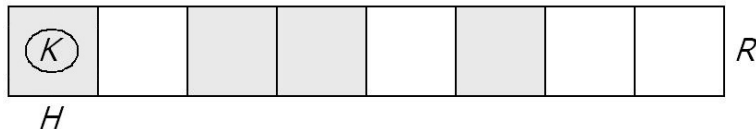
$$Ks \ (s \in S) \text{ is a right coset} \Leftrightarrow \exists t \in S : Kst = K.$$

- ▶ The **(right) translational index** of K is the number of right cosets.

Green's relations interpretation

H - the \mathcal{H} -class of S containing K , R - the \mathcal{R} -class containing K .

- ▶ The right cosets of K partition R
- ▶ K has finite translational index iff $[H : K] < \infty$ and R contains only finitely many \mathcal{H} -classes.



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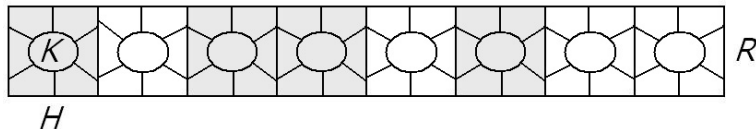
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Properties inherited

S - monoid, K - subgroup of S with finite translational index

Theorem (Ruskuc (1999))

If S is finitely presented (resp. generated) then K is finitely presented (resp. generated).

- ▶ Steinberg (2003) - for inverse semigroups gives an alternative proof using a topological approach.

Higher dimensions - Finite derivation type

- ▶ Is a property of finitely presented semigroups
- ▶ Can be thought of as a higher dimensional version of the property of being finitely presented. Think “relations between relations”
- ▶ Originated from work of C. Squier on finite complete rewriting systems

Theorem (RG & Malheiro (2007))

If S has finite derivation type then K has finite derivation type.

The converse

Cosets are local

- ▶ In general the properties of a single subgroup K of finite index in S will not influence the properties of S .
- ▶ e.g. Adjoin an extra identity 1 to S , let $K = \{1\}$, and consider $K \leq S^1$.

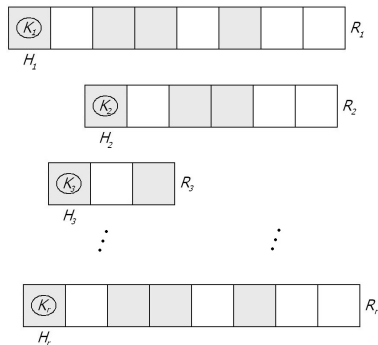
Covering the semigroup

S - monoid, and suppose there exists a finite collection K_1, \dots, K_r of subgroups:

- ▶ each K_i has finite translational index
- ▶ the cosets of $K_1 \dots, K_r$ cover S .

For a given property \mathcal{P} we can ask:

“If all K_i have property \mathcal{P} does it follow that S has property \mathcal{P} ?”



Regular semigroups

Let S be a regular semigroup with finitely many left and right ideals.

Theorem (Ruskuc (1999))

S is finitely presented (resp. finitely generated) if and only if all its maximal subgroups are finitely presented (resp. finitely generated).

Theorem (Golubov (1975))

S is residually finite if and only if all its maximal subgroups are residually finite.

- ▶ Actually, Golubov proved a stronger result, assuming only that each \mathcal{J} -class contains finitely many \mathcal{L} - and \mathcal{R} -classes.

Question What about arbitrary (non-regular) semigroups?

Schützenberger groups

...the groups that never were

Given an \mathcal{H} -class H of any semigroup S we can associate a group with H .

- ▶ Let S be a monoid and let H be any \mathcal{H} -class of S .
- ▶ $T(H) = \{s \in S : Hs = H\}$: the **stabilizer of H** in S .
- ▶ The relation \sim on $T(H)$ defined by:

$$x \sim y \Leftrightarrow (\forall h \in H)(hx = hy)$$

is a congruence.

- ▶ $\Gamma(H) = T(H) / \sim$ is a group, called the **Schützenberger group** of H .

Basic properties

- ▶ $|\Gamma(H)| = |H|$
- ▶ If H_1 and H_2 belong to the same \mathcal{D} -class then $\Gamma(H_1) \cong \Gamma(H_2)$.
- ▶ If H is a group then $\Gamma(H) \cong H$.

Non-regular semigroups

Let S be a monoid with finitely many left and right ideals.

Theorem (Ruskuc (2000))

S is finitely presented (resp. finitely generated) if and only if all its Schützenberger groups are finitely presented (resp. finitely generated).

Theorem (RG & Ruskuc 2007)

S is residually finite if and only if all its Schützenberger groups are residually finite.

- ▶ Applies to non-regular semigroups :-)
- ▶ But does not generalise Golubov :-)

Subsemigroups of monoids: relative Green's relations

Wallace (1962) - developed a theory of relative Green's relations

S - semigroup, T - subsemigroup of S , $a, b \in S$.

Definition (Relative Green's relations)

$$a\mathcal{L}^T b \Leftrightarrow T^1 a = T^1 b, \quad a\mathcal{R}^T b \Leftrightarrow aT^1 = bT^1, \quad \mathcal{H}^T = \mathcal{R}^T \cap \mathcal{L}^T.$$

- ▶ The \mathcal{L}^T -classes are the strong orbits under the action of T^1 on S by left multiplication (dually for \mathcal{R}^T).
- ▶ \mathcal{L}^T , \mathcal{R}^T , and \mathcal{H}^T are equivalence relations.
- ▶ T is a union of \mathcal{R}^T -classes and is a union of \mathcal{L}^T -classes.

Connection with cosets in group theory

G - group, H - subgroup.

- ▶ $\{ \mathcal{R}^H\text{-classes} \} = \{ \text{left cosets of } H \}$
- ▶ $\{ \mathcal{L}^H\text{-classes} \} = \{ \text{right cosets of } H \}$

Green index

S - semigroup, T subsemigroup of S

Definition

The **Green index** of T in S is the number of \mathcal{H}^T -classes of $S \setminus T$.

Example (Finite Green index)

Definition The **Rees index** of T in S is $|S \setminus T|$

- ▶ T has finite Rees index in $S \Rightarrow T$ has finite Green index in S .

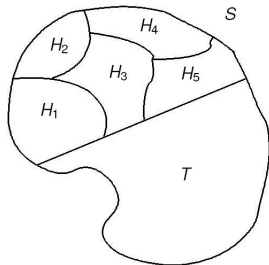
G - group, H - subgroup of G

- ▶ H has finite Green index $\Leftrightarrow H$ has finite group-theoretic index.

Generalised Schützenberger groups

Associated with each \mathcal{H}^T -class H_i is a group Γ_i arising from the action of T on H_i .

Question How are the properties of S related to those of T and the groups Γ_i ?



Green index

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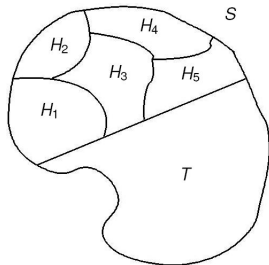
- ▶ T has finite Rees index in $S \Rightarrow T$ has finite Green index in S .

G - group, H - subgroup of G

- ▶ H has finite Green index $\Leftrightarrow H$ has finite group-theoretic index.

- ▶ If T has finite Rees index in S then all the groups Γ_i are finite.
- ▶ If $S = G$ and $T = N \trianglelefteq G$ then $\Gamma_i \cong N$ for all $i \in I$.

In both cases T has $\mathcal{P} \Rightarrow \Gamma_i$ has \mathcal{P} for all i . (\mathcal{P} = any finiteness condition)



Green index results

S - semigroup, T - subsemigroup with finite Green index

H_i ($i \in I$) - relative \mathcal{H} -classes in $S \setminus T$

Γ_i ($i \in I$) - generalised Schützenberger groups

Theorem (RG and Ruskuc (2006))

S is finitely generated (resp. periodic, locally finite) if and only if T is finitely generated (resp. periodic, locally finite), in which case all the groups Γ_i are finitely generated (resp. periodic, locally finite).

Theorem (RG & Ruskuc (2006))

S is residually finite if and only if T and all the groups Γ_i are residually finite.

- ▶ There is an example of a semigroup S , with a finite Green index subsemigroup $T \leq S$ such that T is residually finite but S is not.

Green index results

S - semigroup, T - subsemigroup with finite Green index.

Theorem (RG & Ruskuc (2006))

If T is finitely presented and each group Γ_i is finitely presented then S is finitely presented.

- ▶ These results provide common generalisations of the corresponding results for finite index subgroups of groups, and finite Rees index subsemigroups of semigroups.
- ▶ e.g. for finite generation we obtain a common generalisation of [Schreier's lemma for groups](#) and [Jura \(1978\) for semigroups](#).

Open problem. Prove that if S is finitely presented then T is finitely presented and each group Γ_i is finitely presented.