Approaching cosets using Green's relations and Schützenberger groups

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General question

How are the properties of a semigroup related to those of its substructures?

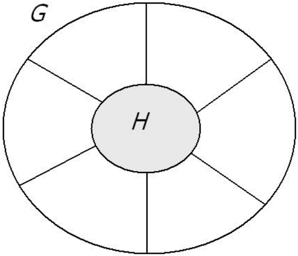
Index in group theory

G - group, H - subgroup of G

- The right cosets of H in G are the sets $\{Hg : g \in G\}$.
- The index $[G : H]$ is the number of (right) cosets of H in G.

G is the disjoint union of the cosets of H

Subgroups of finite index share many properties with their parent groups.



Finiteness conditions

Definition

A property $\mathcal{P}$ of semigroups is a finiteness condition if all finite semigroups satisfy $\mathcal{P}$.

Examples

For a semigroup S the properties of being:

- finite
- finitely generated
- finitely presented
  - S is definable by a presentation $\langle A|R\rangle$ with A and R both finite
- residually finite
  - for any two distinct elements $x, y \in S$ there exists a finite monoid T and a homomorphism $\phi : S \to T$ such that $\phi(x) \neq \phi(y)$

are all finiteness conditions.

And the list goes on: periodic, locally finite, automatic, ...

Finiteness conditions

Proposition

For groups the following finiteness conditions are preserved when taking finite index subgroups and when taking finite index extensions:

- finitely generated
- periodic
- residually finite
- finitely presented
- FP_n
- automatic
- locally finite
- soluble word problem
- finite derivation type.

Idea

- Develop a theory of index for semigroups
- Use it to gain a better understanding of the relationship between the properties of a semigroup and those of its substructures.

Subgroups of monoids: translational index

S - monoid, K - subgroup of S

- The right cosets of K are the elements of the strong orbit of K under the action of S by right multiplication. So

$$Ks \ (s \in S) \text{ is a right coset} \Leftrightarrow \exists t \in S : Kst = K.$$

- The (right) translational index of K is the number of right cosets.

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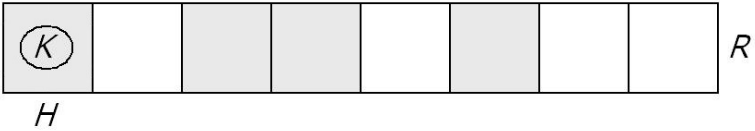
Green's relations interpretation

H - the $\mathcal{H}$-class of S containing K, R - the $\mathcal{R}$-class containing K.

- The right cosets of K partition R
- K has finite translational index iff $[H : K] < \infty$ and R contains only finitely many $\mathcal{H}$-classes.



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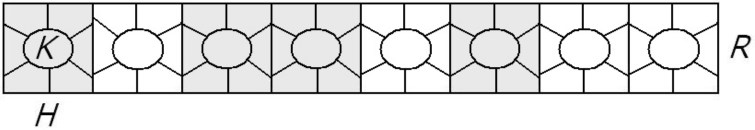

Properties inherited

S - monoid, K - subgroup of S with finite translational index

Theorem (Ruskuc (1999))

If S is finitely presented (resp. generated) then K is finitely presented (resp. generated).

- Steinberg (2003) - for inverse semigroups gives an alternative proof using a topological approach.

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Higher dimensions - Finite derivation type

- Is a property of finitely presented semigroups
- Can be thought of as a higher dimensional version of the property of being finitely presented. Think "relations between relations"
- Originated from work of C. Squier on finite complete rewriting systems

Theorem (RG & Malheiro (2007))

If S has finite derivation type then K has finite derivation type.

The converse

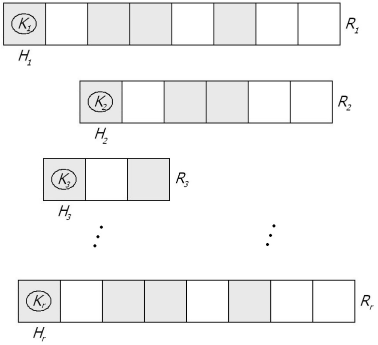
Cosets are local

- In general the properties of a single subgroup K of finite index in S will not influence the properties of S.
- e.g. Adjoin an extra identity 1 to S, let $K = \{1\}$, and consider $K \leq S^1$.

Covering the semigroup

S - monoid, and suppose there exists a finite collection $K_1, \ldots, K_r$ of subgroups:

- each K_i has finite translational index
- the cosets of $K_1 \ldots, K_r$ cover S.



For a given property $\mathcal{P}$ we can ask:

"If all K_i have property $\mathcal{P}$ does it follow that S has property $\mathcal{P}$?"

Regular semigroups

Let S be a regular semigroup with finitely many left and right ideals.

Theorem (Ruskuc (1999))

S is finitely presented (resp. finitely generated) if and only if all its maximal subgroups are finitely presented (resp. finitely generated).

Theorem (RG & Malheiro (2008))

S has finite derivation type if and only if all its maximal subgroups have finite derivation type.

Theorem (Golubov (1975))

S is residually finite if and only if all its maximal subgroups are residually finite.

- Actually, Golubov proved a stronger result, assuming only that each $\mathcal{J}$-class contains finitely many $\mathcal{L}$- and $\mathcal{R}$-classes.

Question What about arbitrary (non-regular) semigroups?

Schützenberger groups

...the groups that never were

Given an $\mathcal{H}$-class H of any semigroup S we can associate a group with H.

- Let S be a monoid and let H be any $\mathcal{H}$-class of S.
- $T(H) = \{s \in S : Hs = H\}$: the stabilizer of H in S.
- The relation $\sim$ on $T(H)$ defined by:

$$x \sim y \Leftrightarrow (\forall h \in H)(hx = hy)$$

  is a congruence.
- $\Gamma(H) = T(H)/\sim$ is a group, called the Schützenberger group of H.

Basic properties

- $|\Gamma(H)| = |H|$
- If H_1 and H_2 belong to the same $\mathcal{D}$-class then $\Gamma(H_1) \cong \Gamma(H_2)$.
- If H is a group then $\Gamma(H) \cong H$.

Non-regular semigroups

Let S be a monoid with finitely many left and right ideals.

Theorem (Ruskuc (2000))

S is finitely presented (resp. finitely generated) if and only if all its Schützenberger groups are finitely presented (resp. finitely generated).

Theorem (RG & Ruskuc 2007)

S is residually finite if and only if all its Schützenberger groups are residually finite.

- Applies to non-regular semigroups :-)
- But does not generalise Golubov :-(

Subsemigroups of monoids: relative Green's relations

Wallace (1962) - developed a theory of relative Green's relations

S - semigroup, T - subsemigroup of S, $a, b \in S$.

Definition (Relative Green's relations)

$a\mathcal{L}^T b \Leftrightarrow T^1 a = T^1 b, \qquad a\mathcal{R}^T b \Leftrightarrow aT^1 = bT^1, \qquad \mathcal{H}^T = \mathcal{R}^T \cap \mathcal{L}^T.$

- The $\mathcal{L}^T$-classes are the strong orbits under the action of T^1 on S by left multiplication (dually for $\mathcal{R}^T$).
- $\mathcal{L}^T$, $\mathcal{R}^T$, and $\mathcal{H}^T$ are equivalence relations.
- T is a union of $\mathcal{R}^T$-classes and is a union of $\mathcal{L}^T$-classes.

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Connection with cosets in group theory

G - group, H - subgroup.

- { $\mathcal{R}^H$-classes } = { left cosets of H }
- { $\mathcal{L}^H$-classes } = { right cosets of H }

Green index

S - semigroup, T subsemigroup of S

Definition

The Green index of T in S is the number of $\mathcal{H}^T$-classes of $S \setminus T$.

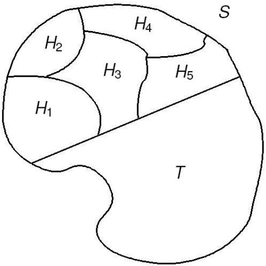
Example (Finite Green index)

Definition The Rees index of T in S is $|S \setminus T|$

- T has finite Rees index in $S \Rightarrow T$ has finite Green index in S.

G - group, H - subgroup of G

- H has finite Green index $\Leftrightarrow H$ has finite group-theoretic index.



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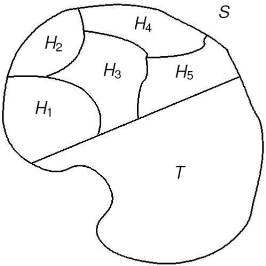
Generalised Schützenberger groups

Associated with each $\mathcal{H}^T$-class H_i is a group Γ_i arising from the action of T on H_i.

Question How are the properties of S related to those of T and the groups Γ_i?



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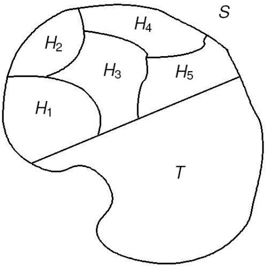
- If T has finite Rees index in S then all the groups Γ_i are finite.
- If $S = G$ and $T = N \trianglelefteq G$ then $\Gamma_i \cong N$ for all $i \in I$.

In both cases T has $\mathcal{P} \Rightarrow \Gamma_i$ has $\mathcal{P}$ for all i.
($\mathcal{P}$ = any finiteness condition)



Green index results

S - semigroup, T - subsemigroup with finite Green index

H_i $(i \in I)$ - relative $\mathcal{H}$-classes in $S \setminus T$

Γ_i $(i \in I)$ - generalised Schützenberger groups

Theorem (RG and Ruskuc (2006))

S is finitely generated (resp. periodic, locally finite) if and only if T is finitely generated (resp. periodic, locally finite), in which case all the groups Γ_i are finitely generated (resp. periodic, locally finite).

Green index results

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Theorem (RG & Ruskuc (2006))

S is residually finite if and only if T and all the groups Γ_i are residually finite.

- There is an example of a semigroup S, with a finite Green index subsemigroup $T \leq S$ such that T is residually finite but S is not.

Green index results

S - semigroup, T - subsemigroup with finite Green index.

Theorem (RG & Ruskuc (2006))

If T is finitely presented and each group Γ_i is finitely presented then S is finitely presented.

- These results provide common generalisations of the corresponding results for finite index subgroups of groups, and finite Rees index subsemigroups of semigroups.
- e.g. for finite generation we obtain a common generalisation of Schreier's lemma for groups and Jura (1978) for semigroups.

Open problem. Prove that if S is finitely presented then T is finitely presented and each group Γ_i is finitely presented.