# Graphs and digraphs with many symmetries

and a wonderfully elegant argument of Hikoe Enomoto

Robert Gray
(joint work with C. E. Praeger and D. Macpherson)

St Andrews Pure Mathematics Colloquium
May 2008

Engineering and Physical Sciences Research Council

# Outline

# Outline

# Homogeneous relational structures

**Definition**

A relational structure $M$ is homogeneous if every isomorphism between finite substructures of $M$ can be extended to an automorphism of $M$.

# Homogeneous relational structures

## Definition

A relational structure $M$ is homogeneous if every isomorphism between finite substructures of $M$ can be extended to an automorphism of $M$.

## Relational structures

- a relational structure consists of a set $A$, and some relations $R_1, \ldots, R_m$ (can be unary, binary, ternary, ...)
- an (induced) substructure is obtained by taking a subset $B \subseteq A$ and keeping only those relations where all entries in the tuple belong to $B$
- an isomorphism is a “structure preserving” mapping (i.e. a bijection $\phi$ such that $\phi$ and $\phi^{-1}$ are both homomorphisms)

## Example

A graph $\Gamma$ is a structure $(V\Gamma, \sim)$ where $V\Gamma$ is a set, and $\sim$ is a symmetric irreflexive binary relation on $V\Gamma$.

# Examples of homogeneous structures

$X$ - a pure set

- automorphism group is the full symmetric group where any partial permutation can be extended to a (full) permutation

$(\mathbb{Q}, \leq)$ - the rationals with their usual ordering

- the automorphisms are the order-preserving permutations
- isomorphisms between finite substructures can be extended to automorphisms that are piecewise-linear

Rado's countable random graph $R$

- if we choose a countable graph at random (edges independently with probability $\frac{1}{2}$), then with probability 1 it is isomorphic to $R$

# Some history

## Origins

- The notion of homogeneous structure goes back to the fundamental work of Fraïssé (1953)
- Fraïssé proved a theorem which helps us determine if a countable structure is homogeneous, using the ideas of:
  - age - the finite substructures they embed, and
  - amalgamation property - the way that they can be glued together

Homogeneous structures are nice because they:

- have “lots of” symmetry;
- often have rich and interesting automorphism groups.

# Connection with model theory

Common theme in model theory:

translation between "model theoretic terminology" and "permutation group theoretic terminology"

# Connection with model theory

Common theme in model theory:

translation between "model theoretic terminology" and "permutation group theoretic terminology"

**Example.**

(I) A structure $M$ is $\aleph_0$-categorical if all countable models of the first-order theory of $M$ are isomorphic to $M$.

(II) A permutation group on an infinite set $\Omega$ is called oligomorphic, if it has finitely many orbits of $n$-tuples, for all $n \geq 1$.

# Connection with model theory

Common theme in model theory:

translation between "model theoretic terminology" and "permutation group theoretic terminology"

**Example.**

(I) A structure $M$ is $\aleph_0$-categorical if all countable models of the first-order theory of $M$ are isomorphic to $M$.

(II) A permutation group on an infinite set $\Omega$ is called oligomorphic, if it has finitely many orbits of $n$-tuples, for all $n \geq 1$.

## Theorem (Ryll-Nardzewski)

*A countable structure $M$ over a first-order language is $\aleph_0$-categorical if and only if $\mathrm{Aut}(M)$ is oligomorphic.*

# Connection with model theory

Common theme in model theory:

translation between "model theoretic terminology" and "permutation group theoretic terminology"

**Example.**

(I) A structure $M$ is $\aleph_0$-categorical if all countable models of the first-order theory of $M$ are isomorphic to $M$.

(II) A permutation group on an infinite set $\Omega$ is called oligomorphic, if it has finitely many orbits of $n$-tuples, for all $n \geq 1$.

**Theorem (Ryll-Nardzewski)**

*A countable structure $M$ over a first-order language is $\aleph_0$-categorical if and only if* $\mathrm{Aut}(M)$ *is oligomorphic.*

Homogeneous structures give examples of "nice" $\aleph_0$-categorical structures (precisely those that have quantifier elimination).

# Classification results

For certain families of relational structure, those members that are homogeneous have been completely determined.

## Some classification results

| | Finite | Countably infinite |
|---|---|---|
| Posets | (trivial) | Schmerl (1979) |
| Tournaments | Woodrow (1976) | Lachlan (1984) |
| Graphs | Gardiner (1976) | Lachlan & Woodrow (1980) |
| Digraphs | Lachlan (1982) | Cherlin (1998) |

# Outline

# Set-homogeneity

## Definition

A relational structure $M$ is set-homogeneous if whenever two finite substructures $U$ and $V$ are isomorphic, there is an automorphism $g \in \mathrm{Aut}(M)$ such that $Ug = V$.

# Set-homogeneity

**Definition**

A relational structure $M$ is set-homogeneous if whenever two finite substructures $U$ and $V$ are isomorphic, there is an automorphism $g \in \mathrm{Aut}(M)$ such that $Ug = V$.

- It is a concept originally due to Fraïssé and Pouzet.
- The permutation group-theoretic weakening

  homogeneous $\rightsquigarrow$ set-homogeneous

  relates to the model-theoretic weakening

  elimination of quantifiers $\rightsquigarrow$ model complete.

- Droste et al. (1994) - proved a set-homogeneous analogue of Fraïssé's theorem, where the amalgamation property is replaced by something called the twisted amalgamation property.

# Set-homogeneity vs homogeneity

- Clearly if $M$ is homogeneous then $M$ is set-homogeneous.
- The converse is not true in general:

## Example

Let $M = (\mathbb{Q}, R)$ where $R$ is the ternary relation given by:

$$\forall x, y, z \in M,\ (x, y, z) \in R \Leftrightarrow x < y < z.$$

- $M$ is set-homogeneous
  - any order-preserving bijection between between finite substructures is an isomorphism that extends to an automorphism

# Set-homogeneity vs homogeneity

- Clearly if $M$ is homogeneous then $M$ is set-homogeneous.
- The converse is not true in general:

## Example

Let $M = (\mathbb{Q}, R)$ where $R$ is the ternary relation given by:

$$\forall x, y, z \in M,\ (x, y, z) \in R \Leftrightarrow x < y < z.$$

- $M$ is set-homogeneous
  - any order-preserving bijection between between finite substructures is an isomorphism that extends to an automorphism
- $M$ is not homogeneous
  - $(0, 1) \mapsto (0, 1)$ is an isomorphism between substructures
  - it does not extend to an automorphism since $(0, \frac{1}{2}, 1) \in R$ but $(1, x, 0) \notin R$ for any $x \in \mathbb{Q}$.

# Set-homogeneity vs homogeneity

- Clearly if $M$ is homogeneous then $M$ is set-homogeneous.
- The converse is not true in general:

## Example

Let $M = (\mathbb{Q}, R)$ where $R$ is the ternary relation given by:

$$\forall x, y, z \in M,\ (x, y, z) \in R \Leftrightarrow x < y < z.$$

- $M$ is set-homogeneous
  - any order-preserving bijection between between finite substructures is an isomorphism that extends to an automorphism
- $M$ is not homogeneous
  - $(0, 1) \mapsto (0, 1)$ is an isomorphism between substructures
  - it does not extend to an automorphism since $(0, \frac{1}{2}, 1) \in R$ but $(1, x, 0) \notin R$ for any $x \in \mathbb{Q}$.

## General question

How much stronger is homogeneity than set-homogeneity?

# Set-homogeneous finite graphs

## Ronse (1978)

...proved that for finite graphs homogeneity and set-homogeneity are equivalent.

- He did this by classifying the finite set-homogeneus graphs and then observing that they are all, in fact, homogeneous.
- This generalised an earlier result of Gardiner, classifying the finite homogeneous graphs.

# Set-homogeneous finite graphs

## Ronse (1978)

...proved that for finite graphs homogeneity and set-homogeneity are equivalent.

- He did this by classifying the finite set-homogeneus graphs and then observing that they are all, in fact, homogeneous.
- This generalised an earlier result of Gardiner, classifying the finite homogeneous graphs.

## Enomoto (1981)

...gave a direct proof of the fact that for finite graphs set-homogeneous implies homogeneous.

- this avoids the need to classify the set-homogeneous graphs
- the set-homogeneous classification can then be read off from Gardiner's result

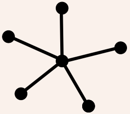
# Some graph theoretic terminology and notation

## Definition

$\Gamma = (V\Gamma, \sim)$ - a graph

So $\sim$ is a symmetric irreflexive binary relation on $V\Gamma$

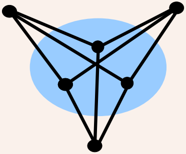
- Let $v$ be a vertex of $\Gamma$. Then the neighbourhood $\Gamma(v)$ of $v$ is the set of all vertices adjacent to $v$. So



$$\Gamma(v) = \{w \in V\Gamma : w \sim v\}$$

- For $X \subseteq V\Gamma$ we define

$$\Gamma(X) = \{w \in V\Gamma : w \sim x \; \forall x \in X\}$$



# Enomoto's argument

**Lemma (Enomoto's lemma)**

*Let* $\Gamma$ *be a finite set-homogeneous graph and let* $U$ *and* $V$ *be induced subgraphs of* $\Gamma$*. If* $U \cong V$ *then* $|\Gamma(U)| = |\Gamma(V)|$.

**Proof.**

# Enomoto's argument

## Lemma (Enomoto's lemma)

*Let $\Gamma$ be a finite set-homogeneous graph and let $U$ and $V$ be induced subgraphs of $\Gamma$. If $U \cong V$ then $|\Gamma(U)| = |\Gamma(V)|$.*

**Proof.**

- Let $g \in \mathrm{Aut}(\Gamma)$ such that $Ug = V$.

# Enomoto's argument

## Lemma (Enomoto's lemma)

*Let* $\Gamma$ *be a finite set-homogeneous graph and let* $U$ *and* $V$ *be induced subgraphs of* $\Gamma$*. If* $U \cong V$ *then* $|\Gamma(U)| = |\Gamma(V)|$.

**Proof.**

- Let $g \in \mathrm{Aut}(\Gamma)$ such that $Ug = V$.
- Then $(\Gamma(U))g = \Gamma(V)$.

# Enomoto's argument

**Lemma (Enomoto's lemma)**

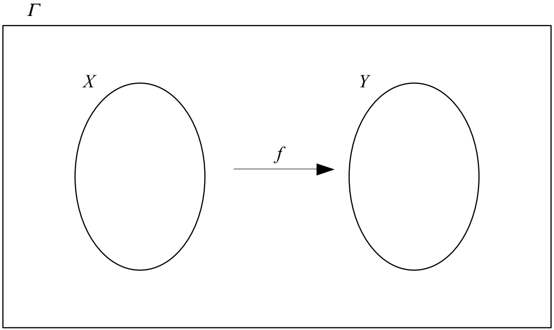
*Let $\Gamma$ be a finite set-homogeneous graph and let $U$ and $V$ be induced subgraphs of $\Gamma$. If $U \cong V$ then $|\Gamma(U)| = |\Gamma(V)|$.*

**Proof.**

- Let $g \in \mathrm{Aut}(\Gamma)$ such that $Ug = V$.
- Then $(\Gamma(U))g = \Gamma(V)$.
- In particular $|\Gamma(U)| = |\Gamma(V)|$.

# Enomoto's argument

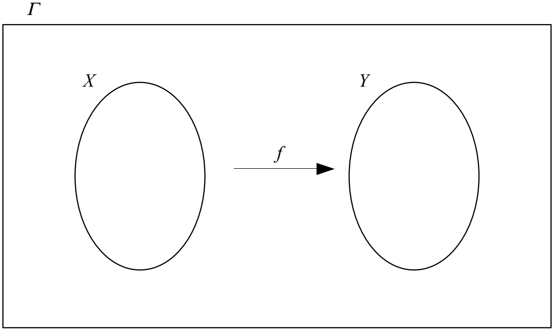
$\Gamma$ - finite set-homogeneous graph   $X, Y$ - induced subgraphs
$f : X \to Y$ an isomorphism




**Claim:** The isomorphism $f : X \to Y$ is either an automorphism, or extends to an isomorphism $f' : X' \to Y'$ where $X' \supsetneq X$ and $Y' \supsetneq Y$.

# Enomoto's argument

$\Gamma$ - finite set-homogeneous graph $\quad X, Y$ - induced subgraphs
$f : X \to Y$ an isomorphism

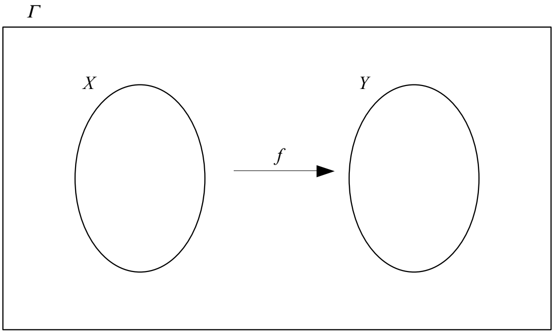



**Proof of claim.**

- Choose $a \in \Gamma \setminus X$ with $|\Gamma(a) \cap X|$ as large as possible.

# Enomoto's argument

$\Gamma$ - finite set-homogeneous graph $X, Y$ - induced subgraphs
$f : X \to Y$ an isomorphism

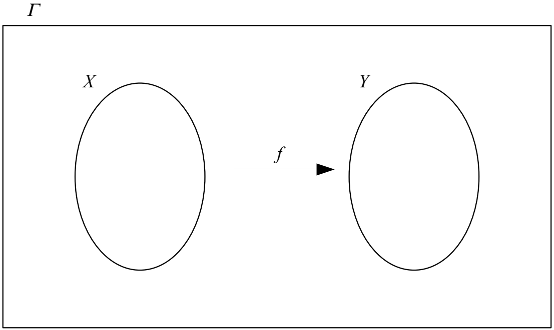



**Proof of claim.**

- Choose $a \in \Gamma \setminus X$ with $|\Gamma(a) \cap X|$ as large as possible.
- Choose $d \in \Gamma \setminus Y$ with $|\Gamma(d) \cap Y|$ as large as possible.

# Enomoto's argument

$\Gamma$ - finite set-homogeneous graph $\quad X, Y$ - induced subgraphs
$f : X \to Y$ an isomorphism

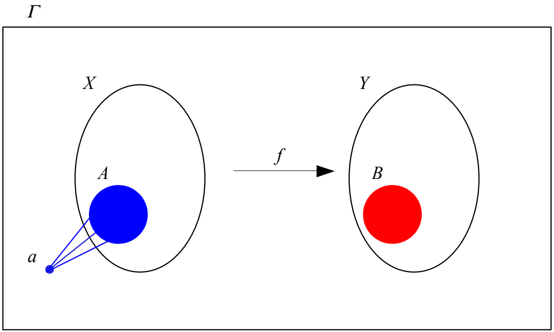



**Proof of claim.**

- Choose $a \in \Gamma \setminus X$ with $|\Gamma(a) \cap X|$ as large as possible.
- Choose $d \in \Gamma \setminus Y$ with $|\Gamma(d) \cap Y|$ as large as possible.
- Suppose $|\Gamma(a) \cap X| \geq |\Gamma(d) \cap Y|$ (the other possibility is dealt with dually using the isomorphism $f^{-1}$)

# Enomoto's argument

$\Gamma$ - finite set-homogeneous graph $\quad X, Y$ - induced subgraphs
$f : X \to Y$ an isomorphism



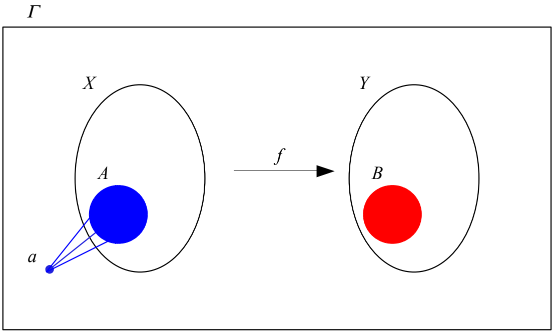

**Proof of claim.**

- Let $A = \Gamma(a) \cap X$ and define $B = f(A)$.

# Enomoto's argument

$\Gamma$ - finite set-homogeneous graph $\quad X, Y$ - induced subgraphs

$f : X \to Y$ an isomorphism

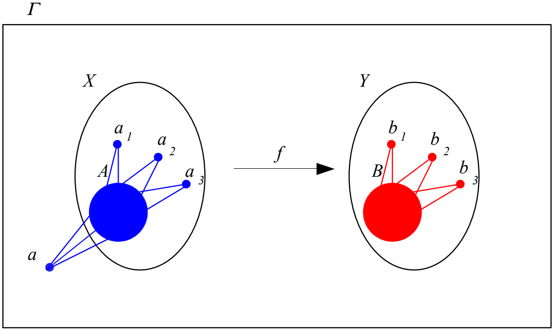



**Proof of claim.**

- Let $A = \Gamma(a) \cap X$ and define $B = f(A)$.
- $A \cong B$ & $\Gamma$ is set-homogeneous so by the lemma $|\Gamma(A)| = |\Gamma(B)|$.

# Enomoto's argument

$\Gamma$ - finite set-homogeneous graph $\quad X, Y$ - induced subgraphs
$f : X \to Y$ an isomorphism

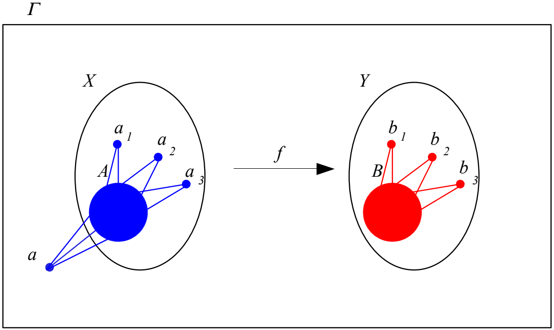



**Proof of claim.**

- Let $A = \Gamma(a) \cap X$ and define $B = f(A)$.
- $A \cong B$ & $\Gamma$ is set-homogeneous so by the lemma $|\Gamma(A)| = |\Gamma(B)|$.
- $\Gamma(B) \cap Y = f(\Gamma(A) \cap X)$ so $|\Gamma(B) \cap Y| = |\Gamma(A) \cap X|$.

# Enomoto's argument

$\Gamma$ - finite set-homogeneous graph $\quad X, Y$ - induced subgraphs
$f : X \to Y$ an isomorphism

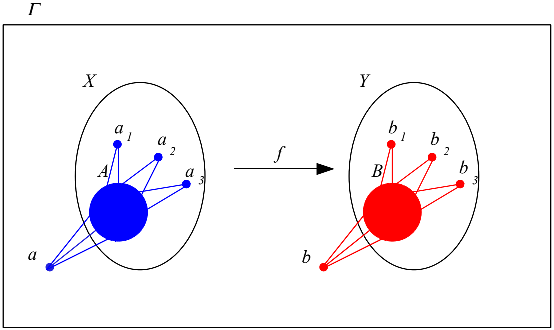



**Proof of claim.**

- Let $A = \Gamma(a) \cap X$ and define $B = f(A)$.
- $A \cong B$ & $\Gamma$ is set-homogeneous so by the lemma $|\Gamma(A)| = |\Gamma(B)|$.
- $\Gamma(B) \cap Y = f(\Gamma(A) \cap X)$ so $|\Gamma(B) \cap Y| = |\Gamma(A) \cap X|$.
- $\therefore |\Gamma(B) \setminus Y| = |\Gamma(A) \setminus X)| \geq 1$

# Enomoto's argument

$\Gamma$ - finite set-homogeneous graph $\quad X, Y$ - induced subgraphs
$f : X \to Y$ an isomorphism

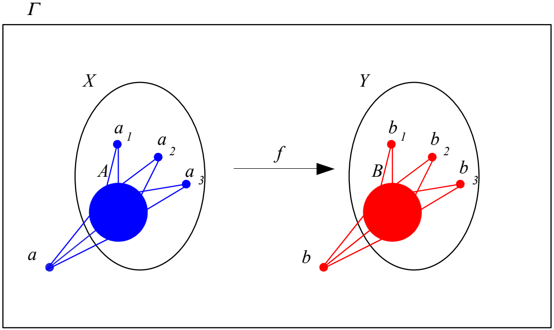



**Proof of claim.**

- Let $b \in \Gamma(B) \setminus Y$ and extend $f$ to $f' : X \cup \{a\} \to Y \cup \{b\}$ by defining $f'(a) = b$.

# Enomoto's argument

$\Gamma$ - finite set-homogeneous graph $\quad X, Y$ - induced subgraphs
$f : X \to Y$ an isomorphism

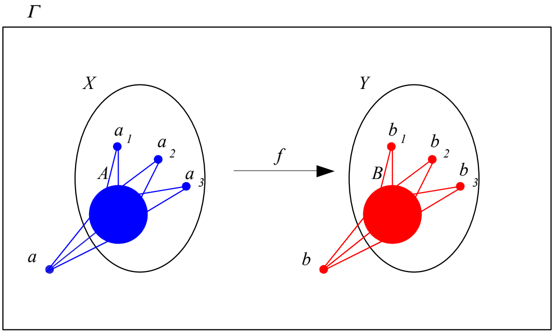



**Proof of claim.**

- Let $b \in \Gamma(B) \setminus Y$ and extend $f$ to $f' : X \cup \{a\} \to Y \cup \{b\}$ by defining $f'(a) = b$.
- $\Gamma(b) \cap Y = B$ by maximality in original definition of $a$,

# Enomoto's argument

$\Gamma$ - finite set-homogeneous graph $\quad X, Y$ - induced subgraphs
$f : X \to Y$ an isomorphism




**Proof of claim.**

- Let $b \in \Gamma(B) \setminus Y$ and extend $f$ to $f' : X \cup \{a\} \to Y \cup \{b\}$ by defining $f'(a) = b$.
- $\Gamma(b) \cap Y = B$ by maximality in original definition of $a$,
- $\therefore f'$ is an isomorphism.

# Set-homogeneous digraphs

**Question:** Does Enomoto's argument apply to other kinds of structure?

# Set-homogeneous digraphs

**Question:** Does Enomoto's argument apply to other kinds of structure?

## Definition (Digraphs)

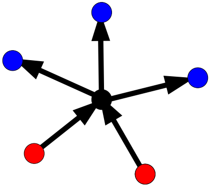
A digraph $D$ consists of a set $VD$ of vertices together with an irreflexive antisymmetric binary relation $\to$ on $VD$.

## Definition (in- and out-neighbours)

A vertex $v \in VD$ has a set of in-neighbours and a set of out-neighbours

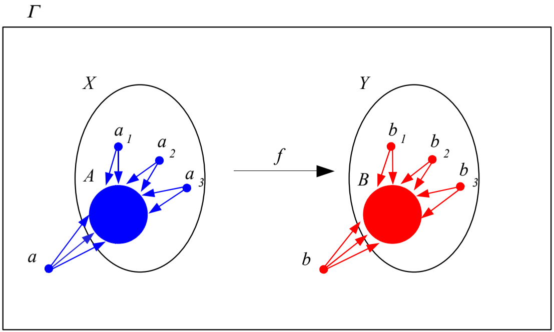
$$D^+(v) = \{w \in VD : v \to w\}, \quad D^-(v) = \{w \in VD : w \to v\}.$$

A vertex with red in-neighbours and blue out-neighbours



# Enomoto's argument for digraphs

$D$ - finite set-homogeneous digraph $\quad X, Y$ - induced subdigraphs
$f : X \to Y$ an isomorphism

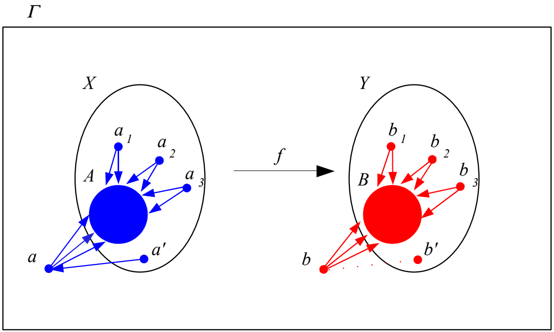



- Follow the same steps but using out-neighbours instead of neighbours.
- Everything works, except the very last step.

# Enomoto's argument for digraphs

$D$ - finite set-homogeneous digraph $\quad X, Y$ - induced subdigraphs

$f : X \to Y$ an isomorphism




- Follow the same steps but using out-neighbours instead of neighbours.
- Everything works, except the very last step.
- We do not know how $b$ is related to the vertices in the set $Y \setminus B$.
  So $f'$ may not be an isomorphism.

# Enomoto's argument for digraphs

**The key point:**

- For graphs, given $u, v \in V\Gamma$ there are 2 possibilities

$$u \sim v \quad \text{or} \quad u \parallel v \ \text{(meaning that } u \,\&\, v \text{ are unrelated).}$$

- For digraphs, given $u, v \in VD$ there are 3 possibilities

$$u \rightarrow v \quad \text{or} \quad v \rightarrow u \quad \text{or} \quad u \parallel v.$$

# Enomoto's argument for digraphs

**The key point:**

- For graphs, given $u, v \in V\Gamma$ there are 2 possibilities

$$u \sim v \quad \text{or} \quad u \parallel v \;\; \text{(meaning that } u \; \& \; v \text{ are unrelated)}.$$

- For digraphs, given $u, v \in VD$ there are 3 possibilities

$$u \longrightarrow v \quad \text{or} \quad v \longrightarrow u \quad \text{or} \quad u \parallel v.$$

**However, the argument does work for tournaments:**

## Definition

A tournament is a digraph where for any pair of vertices $u, v$ either $u \longrightarrow v$ or $v \longrightarrow u$.

## Corollary

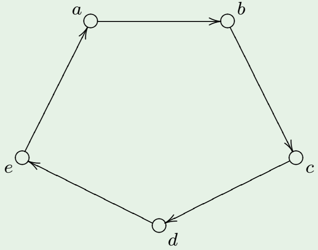
*Let $T$ be a finite tournament. Then $T$ is homogeneous if and only if $T$ is set-homogeneous.*

# A non-homogeneous example

## Example

Let $D_n$ denote the digraph with vertex set $\{0, \ldots, n-1\}$ and just with arcs $i \to i+1 \pmod{n}$.

The digraph $D_5$ is set-homogeneous but is not homogeneous.




- $(a, c) \mapsto (a, d)$ gives an isomorphism between induced subdigraphs that does not extend to an automorphism
- However, there is an automorphism sending $\{a, c\}$ to $\{a, d\}$.

# Finite set-homogeneous digraphs

**Question**

How much bigger is the class of set-homogeneous digraphs than the class of homogeneous digraphs?

# Finite set-homogeneous digraphs

### Question

How much bigger is the class of set-homogeneous digraphs than the class of homogeneous digraphs?

### Theorem (RG, Macpherson, Praeger (2007))

*Let $D$ be a finite set-homogeneous digraph. Then either $D$ is homogeneous or it is isomorphic to $D_5$.*

**Proof.**

- Carry out the classification of finite set-homogeneous digraphs.
- By inspection note that $D_5$ is the only non-homogeneous example. □

# Symmetric-digraphs (s-digraphs)

A common generalisation of graphs and digraphs

## Definition (s-digraph)

- An s-digraph is the same as a digraph except that we allow pairs of vertices to have arcs in both directions.
- So for any pair of vertices $u, v$ exactly one of the following holds:

$$u \to v, \qquad v \to u, \qquad u \leftrightarrow v, \qquad u \parallel v.$$

# Symmetric-digraphs (s-digraphs)

A common generalisation of graphs and digraphs

## Definition (s-digraph)

- An s-digraph is the same as a digraph except that we allow pairs of vertices to have arcs in both directions.
- So for any pair of vertices $u, v$ exactly one of the following holds:

$$u \longrightarrow v, \qquad v \longrightarrow u, \qquad u \leftrightarrow v, \qquad u \parallel v.$$

- Formally we can think of an s-digraph as a structure $M$ with two binary relations $\longrightarrow$ and $\sim$ where
  - $\sim$ is irreflexive and symmetric (and corresponds to $\leftrightarrow$ above)
  - $\longrightarrow$ is irreflexive and antisymmetric
  - $\sim$ and $\longrightarrow$ are disjoint
- A graph is an s-digraph (where there are no $\longrightarrow$-related vertices)
- A digraph is an s-digraph (where there are no $\sim$-related vertices)

# Classifying the finite homogeneous s-digraphs

- Lachlan (1982) classified the finite homogeneous s-digraphs

To state his result we need the notions of

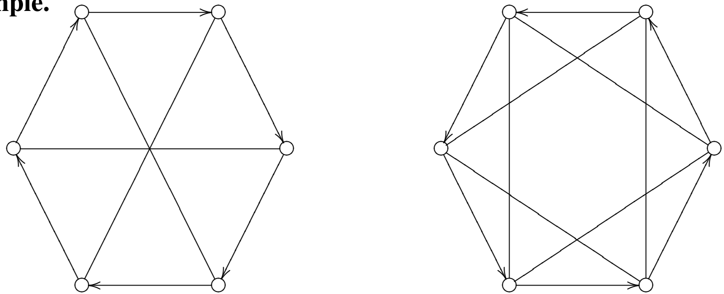
- complement
- compositional product

# Finite homogeneous s-digraphs

## Definition (Complement)

If $M$ is an s-digraph, then $\bar{M}$, the complement, is the s-digraph with the same vertex set, such that $u \sim v$ in $\bar{M}$ if and only if they are unrelated in $M$, and $u \to v$ in $\bar{M}$ if and only if $v \to u$ in $M$.

**Example.**



$M$ $\qquad\qquad$ $\overline{M}$

# Finite homogeneous s-digraphs

## Definition (Composition)

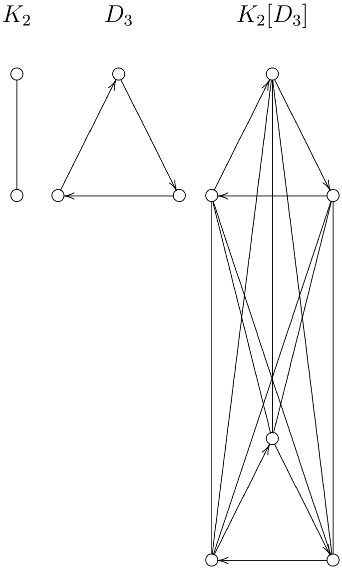
If $U$ and $V$ are s-digraphs, the compositional product $U[V]$ denotes the s-digraph which is

"$|U|$ copies of $V$"

Vertex set = $U \times V$

$\rightarrow$ relations are of form $(u, v_1) \rightarrow (u, v_2)$ where $v_1 \rightarrow v_2$ in $V$, or of form $(u_1, v_1) \rightarrow (u_2, v_2)$ where $u_1 \rightarrow u_2$ in $U$,

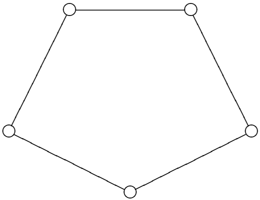
Similarly for $\sim$.



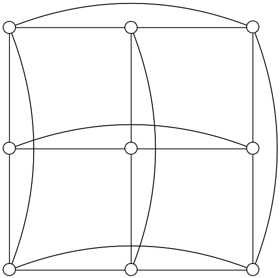
# Some finite homogeneous s-digraphs

## Sporadic examples

$\mathcal{L}$ - finite homogeneous graphs, $\mathcal{A}$ - finite homogeneous digraphs,
$\mathcal{S}$ - finite homogeneous s-digraphs



$C_5 \in \mathcal{L}$

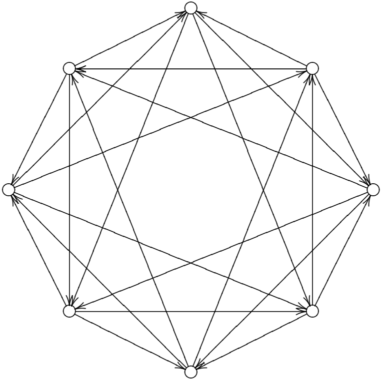


$K_3 \times K_3 \in \mathcal{L}$

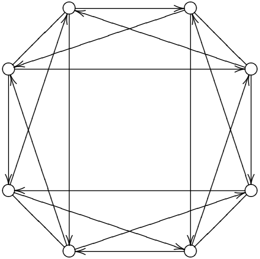
# Some finite homogeneous s-digraphs

## Sporadic examples

$\mathcal{L}$ - finite homogeneous graphs, $\mathcal{A}$ - finite homogeneous digraphs,
$\mathcal{S}$ - finite homogeneous s-digraphs



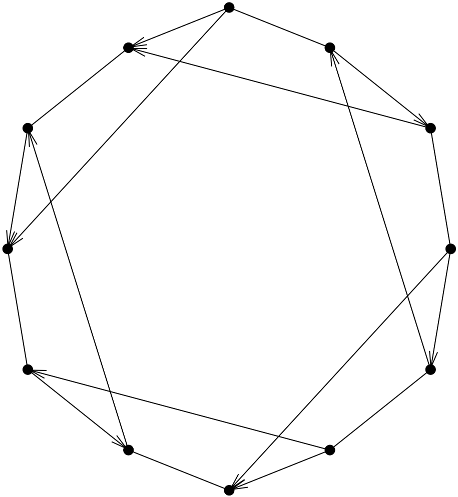
$H_0 \in \mathcal{A}$



$H_1 \in \mathcal{S}$

# Some finite homogeneous s-digraphs

## Sporadic examples



$H_2 \in \mathcal{S}$

To complete the picture...

In $H_2$ each vertex $v$ has a unique mate $v'$ to which it is joined by an undirected edge.

Now if $v \to w$ then $w \to v'$ where $v'$ is the mate of $v$.

Similarly, if $w \to v$ then $v' \to w$.

# Lachlan's classification

$\mathcal{L}$ - finite homogeneous graphs, $\mathcal{A}$ - finite homogeneous digraphs,
$\mathcal{S}$ - finite homogeneous s-digraphs

## Theorem (Lachlan (1982))

*Let $M$ be a finite s-digraph. Then*

***Gardiner***

*(i) $M \in \mathcal{L} \Leftrightarrow M$ or $\bar{M}$ is one of: $C_5$, $K_3 \times K_3$, $K_m[\bar{K}_n]$ (for $1 \leq m, n \in \mathbb{N}$);*

# Lachlan's classification

$\mathcal{L}$ - finite homogeneous graphs, $\mathcal{A}$ - finite homogeneous digraphs,
$\mathcal{S}$ - finite homogeneous s-digraphs

## Theorem (Lachlan (1982))

*Let $M$ be a finite s-digraph. Then*

***Gardiner***

*(i) $M \in \mathcal{L} \Leftrightarrow M$ or $\bar{M}$ is one of: $C_5$, $K_3 \times K_3$, $K_m[\bar{K}_n]$ (for $1 \leq m, n \in \mathbb{N}$);*

***Lachlan***

*(ii) $M \in \mathcal{A} \Leftrightarrow M$ is one of: $D_3$, $D_4$, $H_0$, $\bar{K}_n$, $\bar{K}_n[D_3]$, or $D_3[\bar{K}_n]$, for some $n \in \mathbb{N}$ with $1 \leq n$;*

# Lachlan's classification

$\mathcal{L}$ - finite homogeneous graphs, $\mathcal{A}$ - finite homogeneous digraphs,
$\mathcal{S}$ - finite homogeneous s-digraphs

## Theorem (Lachlan (1982))

*Let $M$ be a finite s-digraph. Then*

***Gardiner***

*(i) $M \in \mathcal{L} \Leftrightarrow M$ or $\bar{M}$ is one of: $C_5$, $K_3 \times K_3$, $K_m[\bar{K}_n]$ (for $1 \leq m, n \in \mathbb{N}$);*

***Lachlan***

*(ii) $M \in \mathcal{A} \Leftrightarrow M$ is one of: $D_3$, $D_4$, $H_0$, $\bar{K}_n$, $\bar{K}_n[D_3]$, or $D_3[\bar{K}_n]$, for some $n \in \mathbb{N}$ with $1 \leq n$;*

*(iii) $M \in \mathcal{S} \Leftrightarrow M$ or $\bar{M}$ is isomorphic to an s-digraph of one of the following forms. $K_n[A], A[K_n]$, $L$, $D_3[L]$, $L[D_3]$, $H_1$, $H_2$, where $n \in \mathbb{N}$ with $1 \leq n$, $A \in \mathcal{A}$ and $L \in \mathcal{L}$.*

# Set-homogeneous s-digraphs

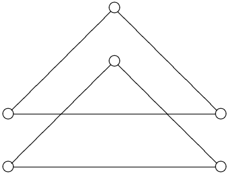
## Theorem (RG, Macpherson, Praeger (2007))

*The finite s-digraphs that are set-homogeneous but not homogeneous are:*

**Infinite families (with $n \in \mathbb{N}$)**

(i) $K_n[D_5]$ or $D_5[K_n]$

(ii) $J_n$



$J_2$

# Set-homogeneous s-digraphs

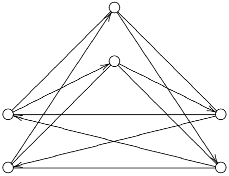
## Theorem (RG, Macpherson, Praeger (2007))

*The finite s-digraphs that are set-homogeneous but not homogeneous are:*

**Infinite families (with $n \in \mathbb{N}$)**

(i) $K_n[D_5]$ or $D_5[K_n]$

(ii) $J_n$



$J_2$

# Set-homogeneous s-digraphs

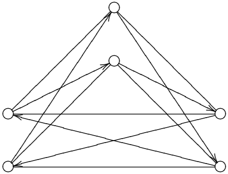
Theorem (RG, Macpherson, Praeger (2007))

*The finite s-digraphs that are set-homogeneous but not homogeneous are:*

**Infinite families (with $n \in \mathbb{N}$)**

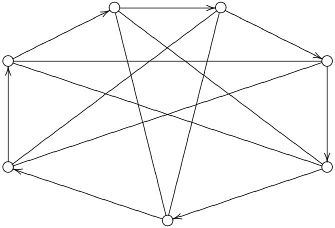
(i) $K_n[D_5]$ or $D_5[K_n]$

(ii) $J_n$

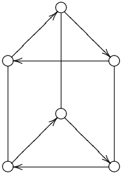


$J_2$

**Sporadics**



$E_7$



$F_6$

# Outline

# Circular structures

- Construction discovered independently by Cameron and Macpherson.

## Definition (the graph $R(3)$)

- its vertex set is any countable dense subset of the unit circle such that no two points make an angle of $2\pi/3$ at the centre of the circle.

# Circular structures

- Construction discovered independently by Cameron and Macpherson.

## Definition (the graph $R(3)$)

- its vertex set is any countable dense subset of the unit circle such that no two points make an angle of $2\pi/3$ at the centre of the circle.
  - To construct such a set begin with the set of all complex roots of unity
  - partition into sets of size 3 with two vertices in the same part iff the angle they make at the centre is a multiple of $2\pi/3$
  - choose representatives from these equivalence classes at random

# Circular structures

- Construction discovered independently by Cameron and Macpherson.

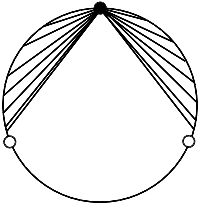
## Definition (the graph $R(3)$)

- its vertex set is any countable dense subset of the unit circle such that no two points make an angle of $2\pi/3$ at the centre of the circle.
  - To construct such a set begin with the set of all complex roots of unity
  - partition into sets of size 3 with two vertices in the same part iff the angle they make at the centre is a multiple of $2\pi/3$
  - choose representatives from these equivalence classes at random
- two vertices are adjacent iff the acute angle they make with the centre of the circle is less than $2\pi/3$
- i.e. they are adjacent when close enough together

**Fact.** Two graphs satisfying these properties are isomorphic.

# Countable set-homogeneous graphs

The neighbourhood of a vertex in the graph $R(3)$



**Theorem (Droste, Giraudet, Macpherson, Sauer (1994))**

*The graph $R(3)$ is set-homogeneous but not 3-homogeneous. Moreover, any set-homogeneous but not $\leq 3$-homogeneous graph is isomorphic to $R(3)$ or its complement.*

# $T(4)$: a countable set-homogeneous digraph

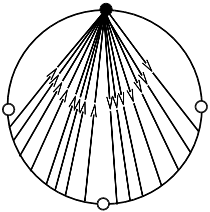
## Definition

Let $T(4)$ be the digraph obtained by distributing countably many points densely around the unit circle

- no two making an angle of $\pi$ or $\pi/2$ at the centre
- such that $x \to y$ if and only if $\pi/2 < \arg(x/y) < \pi$.

By a back-and-forth argument, this construction for $T(4)$ determines unique digraph.

The neighbourhood of a vertex in the graph $T(4)$



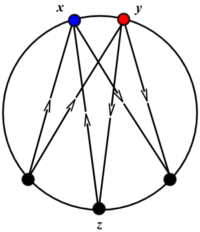
# Properties of $T(4)$

## Lemma

*The digraph $T(4)$ is set-homogeneous but not 2-homogeneous.*

**proof.**

- set-homogeneity: shown by "expanding" $T(4)$ to a homogeneous structure
- not 2-homogeneous: there exist independent pairs that cannot be swapped by any automorphism
- e.g. if $x, y \in T(4)$ with $0 < \arg(x/y) < \pi/2$, then
  - $\exists z(z \to x \wedge y \to z)$ but
  - $\neg\exists z(z \to y \wedge x \to z)$.



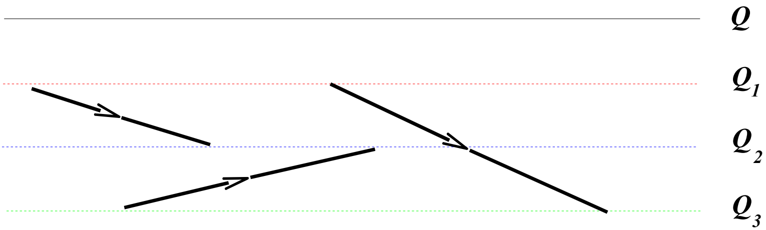
# $R_n$ $(n \geq 2)$: a family of set-homogeneous digraphs

## Definition

- Let $2 \leq n \leq \aleph_0$
- let $\{Q_i : i < n\}$ be a partition of $\mathbb{Q}$ into $n$ dense codense sets.

Define a digraph $R_n$ with domain $\mathbb{Q}$, putting $a \to b$ if and only if $a < b$ and there is no $i < n$ such that $a, b \in Q_i$.

By a back-and-forth argument, this construction for $R_n$ determines unique digraph.



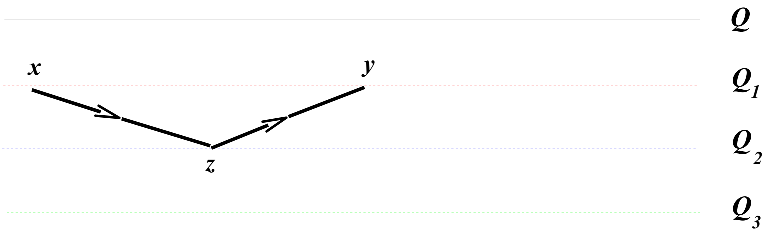
# Properties of $R_n$

**Lemma**

*The digraphs $R_n$ (for $n \geq 2$) are set-homogeneous but not 2-homogeneous.*

**proof.**

- set-homogeneity: shown by "expanding" to a homogeneous structure
- not 2-homogeneous: for if $x, y \in Q_1$ with $x < y$ then there is $z$ with $x \to z \to y$ but no $z$ with $y \to z \to x$.
- $\therefore (x, ymapsto(y, x))$ does not extend to an automorphism.



# A partial classification

## Theorem (RG, Macpherson, Praeger (2007))

*Let $D$ be a countably infinite set-homogeneous digraph which is not 2-homogeneous. Then $D$ is isomorphic to $T(4)$ or to $R_n$ for some $n \geq 2$.*

## Open problems

- Is there a countably infinite tournament that is set-homogeneous but not homogeneous?
- Classify the countably infinite set-homogeneous graphs (and digraphs).

Relating to the first of these questions, we know:

## Proposition (RG, Macpherson, Praeger (2007))

Let $T$ be a set-homogeneous tournament. Then $T$ is 4-homogeneous.