One-relator groups, monoids and inverse monoids

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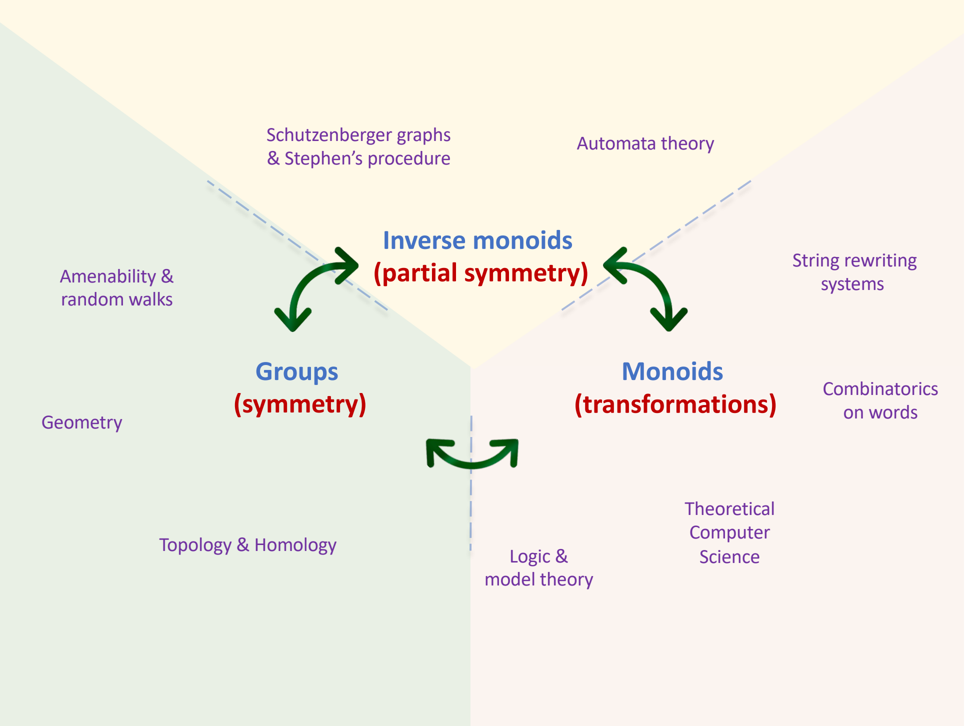
Schutzenberger graphs
& Stephen's procedure
Automata theory
Inverse monoids
(partial symmetry)
Amenability &
random walks
String rewriting
systems
Groups
(symmetry)
Monoids
(transformations)
Geometry
Combinatorics
on words
Theoretical
Computer
Science
Topology & Homology
Logic &
model theory

One-relator monoids

$$\mathrm{Mon}\langle A \mid R\rangle = \mathrm{Mon}\langle \underbrace{a_1, \ldots, a_n}_{\text{letters / generators}} \mid \underbrace{u_1 = v_1, \ldots, u_m = v_m}_{\text{words / defining relations}} \rangle$$

- Defines the monoid $M = A^* / \sim$ where $\sim$ is the equivalence relation with $\alpha \sim \beta$ if α can be transformed into β the other by applying relations R.

Longstanding open problem

Is the word problem decidable for one-relator monoids $\mathrm{Mon}\langle A \mid u = v\rangle$?

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Theorem (Adian & Oganesian, 1978+1987)

The word problem for a given $\mathrm{Mon}\langle A \mid u = v \rangle$ can be reduced to the word problem for a one-relator monoid of the form

$$\mathrm{Mon}\langle a, b \mid bUa = aVa \rangle \quad \text{or} \quad \mathrm{Mon}\langle a, b \mid bUa = a \rangle.$$

- Both of these cases remain open!

Reduction to inverse monoids

- Magnus 1932: One-relator groups have decidable word problem.

The monoids $\mathrm{Mon}\langle a, b \mid bUa = aVa\rangle$ and $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$ are not group embeddable. However Ivanov, Margolis, Meakin (2001) proved that

$$\mathrm{Mon}\langle a, b \mid bUa = aVa\rangle \hookrightarrow \mathrm{Inv}\langle a, b \mid (aVa)^{-1}bUa = 1\rangle \quad \&$$
$$\mathrm{Mon}\langle a, b \mid bUa = a\rangle \hookrightarrow \mathrm{Inv}\langle a, b \mid a^{-1}bUa = 1\rangle.$$

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Theorem (Ivanov, Margolis, Meakin (2001))

If the word problem is decidable for all inverse monoids of the form $\mathrm{Inv}\langle A \mid w = 1\rangle$ then the word problem is also decidable for every one-relator monoid $\mathrm{Mon}\langle A \mid u = v\rangle$.

Word problem for $\mathrm{Inv}\langle A \mid w = 1\rangle$ decidable in many cases:

- Idempotent word [Birget, Margolis, Meakin, 1993, 1994]
- w-strictly positive [Ivanov, Margolis, Meakin, 2001]
- Adjan or Baumslag-Solitar type [Margolis, Meakin, Šuniḱ, 2005]
- Sparse word [Hermiller, Lindblad, Meakin, 2010]

Word problem for one-relator inverse monoids

Theorem (RDG (2020))

There is a one-relator inverse monoid $\mathrm{Inv}\langle A \mid w = 1 \rangle$ with undecidable word problem.

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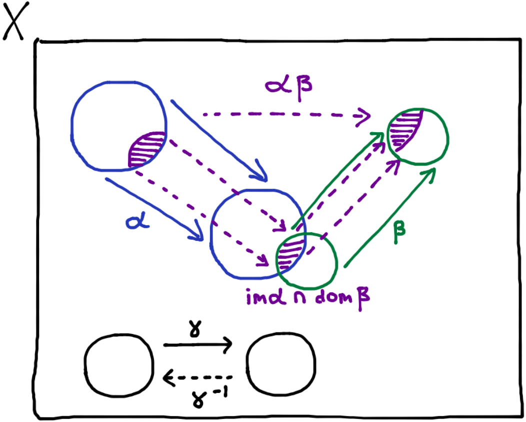
Ingredients for the proof:

- Submonoid membership problem for one relator groups.
- Right-angled Artin groups (RAAGs).
- Right units of inverse monoids and Stephen's procedure for constructing Schützenberger graphs.
- Properties of E-unitary inverse monoids.

Inverse monoids

An inverse monoid is a monoid M such that for every $x \in M$ there is a unique $x^{-1} \in M$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

Example: I_X = monoid of all partial bijections $X \to X$



Examples: In I_3

$$\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & -\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 3 & - & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ - & 1 & -\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & -\end{pmatrix}^{-1} = \begin{pmatrix}1 & 2 & 3\\ - & 1 & 2\end{pmatrix}$$

Note:
$\gamma\gamma^{-1} = \mathrm{id}_{\mathrm{dom}\gamma}$

Inverse monoid presentations

An inverse monoid is a monoid M such that for every $x \in M$ there is a unique $x^{-1} \in M$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

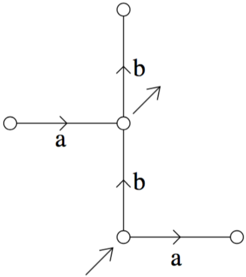
For all $x, y \in M$ we have

$$x = xx^{-1}x, \ (x^{-1})^{-1} = x, \ (xy)^{-1} = y^{-1}x^{-1}, \ xx^{-1}yy^{-1} = yy^{-1}xx^{-1} \qquad (\dagger)$$

$$\mathrm{Inv}\langle A \mid u_i = v_i \ (i \in I)\rangle = \mathrm{Mon}\langle A \cup A^{-1} \mid u_i = v_i \ (i \in I) \cup (\dagger)\rangle$$

where $u_i, v_i \in (A \cup A^{-1})^*$ and x, y range over all words from $(A \cup A^{-1})^*$.

Free inverse monoid $\mathrm{FIM}(A) = \mathrm{Inv}\langle A \mid \ \rangle$



Munn (1974)

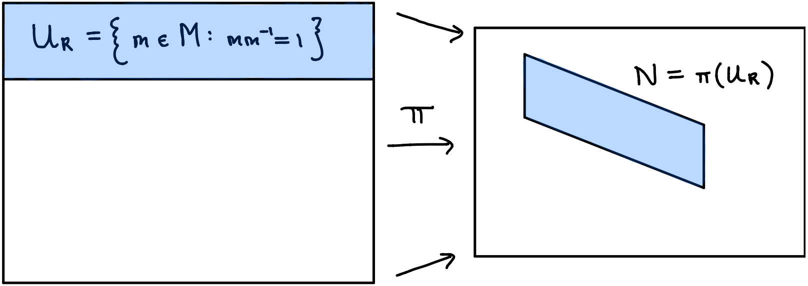
Elements of $\mathrm{FIM}(A)$ can be represented using Munn trees. e.g. in $\mathrm{FIM}(a, b)$ we have $u = w$ where

$u = aa^{-1}bb^{-1}ba^{-1}abb^{-1}$

$w = bbb^{-1}a^{-1}ab^{-1}aa^{-1}b$

Proof strategy

$$M = \mathrm{Inv}\langle A \mid r = 1 \rangle \longrightarrow G = \mathrm{Gp}\langle A \mid r = 1 \rangle$$



If M has decidable word problem

$\Longrightarrow$ membership problem for $U_R \leq M$ is decidable

since for $w \in (A \cup A^{-1})^*$ $\boxed{w \in U_R \iff ww^{-1} = 1}$

(sometimes)

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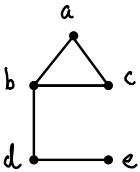
Right-angled Artin groups

Definition

The right-angled Artin group $A(\Gamma)$ associated with the graph Γ is

$$\mathrm{Gp}\langle V\Gamma \mid uv = vu \text{ if and only if } \{u, v\} \in E\Gamma\rangle.$$

Example

$\Gamma =$ 

$$A(\Gamma) = \mathrm{Gp}\langle a, b, c, d, e \mid ac = ca,\ de = ed,\ ab = ba,\ bc = cb,\ bd = db \rangle$$

Submonoid membership problem

G - a finitely generated group with a finite group generating set A.

$\pi : (A \cup A^{-1})^* \to G$ – the canonical monoid homomorphism.

T – a finitely generated submonoid of G.

The membership problem for T within G is decidable if there is an algorithm which solves the following decision problem:

INPUT: A word $w \in (A \cup A^{-1})^*$.

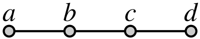
QUESTION: $\pi(w) \in T$?

Theorem (Lohrey & Steinberg (2008))

$A(\Gamma)$ has decidable submonoid membership problem $\Leftrightarrow$ Γ does not embed a square C_4 or a path P_4 with four vertices as an induced subgraph.

HNN-extension of $A(P_4)$ over $A(P_3)$

Let P_4 be the graph



$$A(P_4) = \mathrm{Gp}\langle a, b, c, d \mid ab = ba, bc = cb, cd = dc \rangle.$$

Δ_1 - subgraph induced by $\{a, b, c\}$, Δ_2 subgraph induced by $\{b, c, d\}$,
$\psi : \Delta_1 \to \Delta_2$ - the isomorphism $a \mapsto b$, $b \mapsto c$, and $c \mapsto d$.

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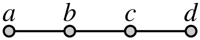

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$$\begin{aligned} & A(P_4, \psi) \\ = \quad & \mathrm{Gp}\langle a, b, c, d, t \mid ab = ba, bc = cb, cd = dc, tat^{-1} = b, tbt^{-1} = c, tct^{-1} = d\rangle \end{aligned}$$

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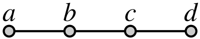

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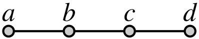

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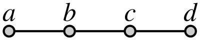

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Conclusion

$A(P_4)$ embeds into the one-relator group

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Right-angled Artin subgroups of one-relator groups

Theorem (RDG (2020))

There is a one-relator group $G = \mathrm{Gp}\langle A \mid r = 1 \rangle$ with a fixed finitely generated submonoid $N \leq G$ such that the membership problem for N within G is undecidable.

Proof:

- Lohrey & Steinberg (2008) proved that $A(P_4)$ contains a finitely generated submonoid T in which membership is undecidable.
- Let $G = \mathrm{Gp}\langle A \mid r = 1 \rangle$ be a one-relator group embedding $\theta : A(P_4) \hookrightarrow G$.
- Then $N = \theta(T)$ is a finitely generated submonoid of G in which membership is undecidable. $\square$

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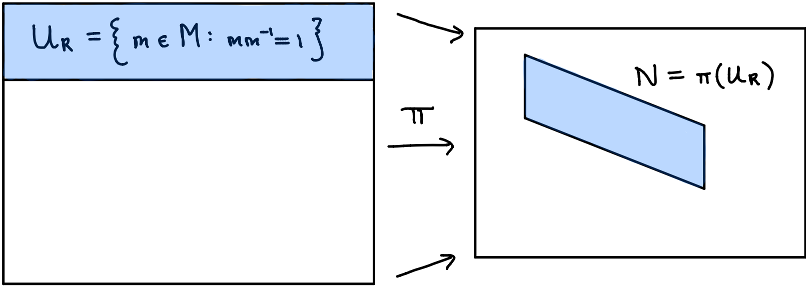
Corollary

$A(\Gamma)$ embeds into some one-relator group $\iff$ Γ is a finite forest.

($\Leftarrow$) Uses Koberda (2013) showing if F is a finite forest $A(F) \hookrightarrow A(P_4)$.
($\Rightarrow$) Uses a result of Louder and Wilton (2017) on Betti numbers of subgroups of torsion-free one-relator groups.

Proof strategy

$$M = \mathrm{Inv}\langle A \mid r = 1\rangle \longrightarrow G = \mathrm{Gp}\langle A \mid r = 1\rangle$$



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Schützenberger graphs

Let $M = \mathrm{Inv}\langle A \mid r = 1\rangle$ and $U_R = \{m \in M : mm^{-1} = 1\}$ the right units of M.

Aim: Construct an $M = \mathrm{Inv}\langle A \mid r = 1\rangle$ such that membership in $U_R \leq M$ is undecidable i.e. it is undecidable whether $uu^{-1} = 1$ for a given $u \in (A \cup A^{-1})^*$. Then M will have undecidable word problem.

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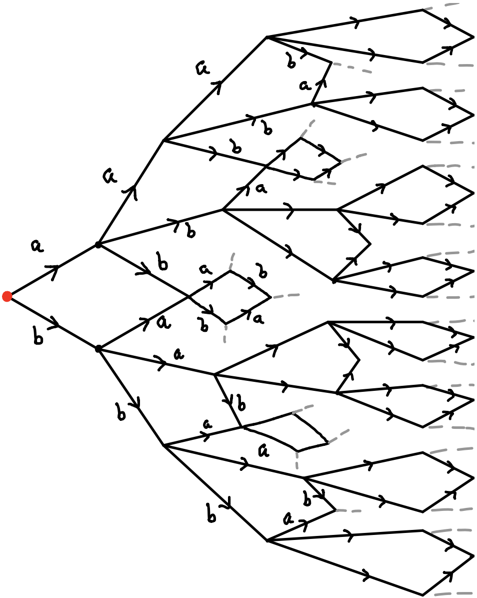
Definition

The Schützenberger graph $S\Gamma(1)$ of $M = \mathrm{Inv}\langle A \mid r = 1\rangle$ is the subgraph of the Cayley graph of M induced on the set of right units of M.

Stephen's procedure

The Schützenberger graph $S\Gamma(1)$ can be obtained as the limit of a sequence of labelled digraphs obtained by an iterative construction given by successively applying operations called expansions and Stallings foldings.

Example - Stephen's Procedure



$\mathrm{Inv}\langle a, b \mid aba^{-1}b^{-1} = 1\rangle$

Stephen's procedure

Expansions: Attach a simple closed path labelled by r at a vertex (if one does not already exist).

Stallings foldings: Identify two edges with the same label and the same initial or terminal vertex.

This process may not stop. Stephen shows that the

- process is confluent &
- limits in an appropriate sense to $S\Gamma(1)$.

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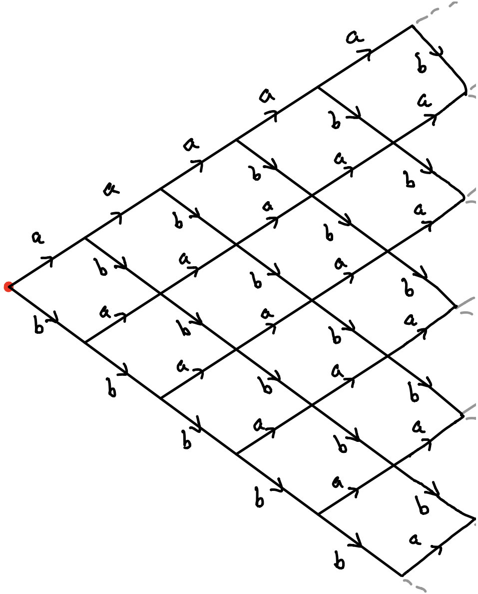

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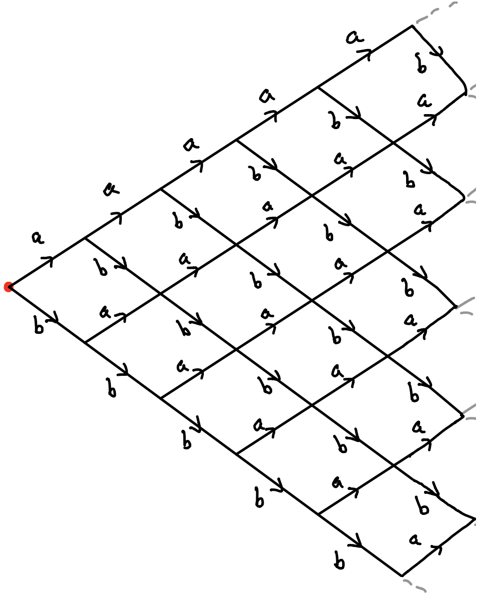
Right unit membership



$\mathrm{Inv}\langle a, b \mid aba^{-1}b^{-1} = 1 \rangle$

$w \in (A \cup A^{-1})^*$ is a right unit $\Leftrightarrow$ w can be read from the origin in $S\Gamma(1)$.

Examples

$aaba^{-1}a^{-1}$ is a right unit.

Note: This word cannot be read in the previous unfolded graph.

$bab^{-1}b^{-1}a$ is not a right unit.

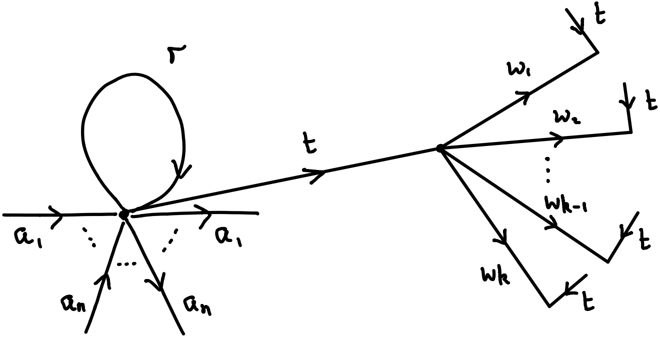
A general construction

For any $r, w_1, \ldots w_k \in (A \cup A^{-1})^*$, with $A = \{a_1, \ldots, a_n\}$, set e equal to
$a_1 a_1^{-1} \ldots a_n a_n^{-1} (t w_1 t^{-1})(t w_1^{-1} t^{-1})(t w_2 t^{-1})(t w_2^{-1} t^{-1}) \ldots (t w_k t^{-1})(t w_k^{-1} t^{-1}) a_n^{-1} a_n \ldots a_1^{-1} a_1$
where t is a new symbol.

Key claim

Let T be the submonoid of $G = \mathrm{Gp}\langle A \mid r = 1\rangle$ generated by $\{w_1, w_2, \ldots, w_k\}$, and let $M = \mathrm{Inv}\langle A, t \mid er = 1\rangle$. Then for all $u \in (A \cup A^{-1})^*$ we have

$$tut^{-1} \in U_R \text{ in } M \iff u \in T \text{ in } G.$$



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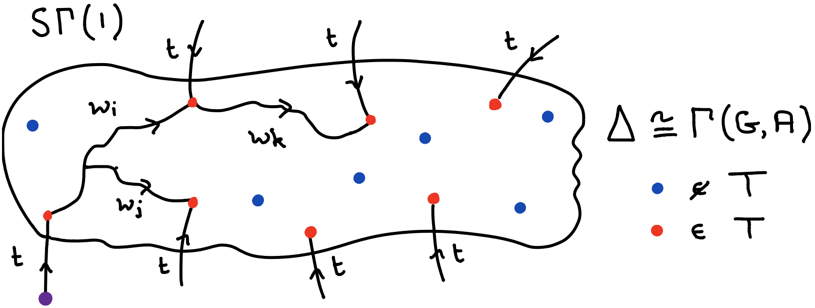

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Theorem (RDG 2020)

If $M = \mathrm{Inv}\langle A, t \mid er = 1\rangle$ has decidable word problem then the membership problem for T within $G = \mathrm{Gp}\langle A \mid r = 1\rangle$ is decidable.

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For any $r, w_1, \ldots w_k \in (A \cup A^{-1})^*$, with $A = \{a_1, \ldots, a_n\}$, set e equal to $a_1 a_1^{-1} \ldots a_n a_n^{-1} (tw_1 t^{-1})(tw_1^{-1} t^{-1})(tw_2 t^{-1})(tw_2^{-1} t^{-1}) \ldots (tw_k t^{-1})(tw_k^{-1} t^{-1}) a_n^{-1} a_n \ldots a_1^{-1} a_1$ where t is a new symbol.

Key claim

Let T be the submonoid of $G = \mathrm{Gp}\langle A \mid r = 1\rangle$ generated by $\{w_1, w_2, \ldots, w_k\}$, and let $M = \mathrm{Inv}\langle A, t \mid er = 1\rangle$. Then for all $u \in (A \cup A^{-1})^*$ we have

$$tut^{-1} \in U_R \text{ in } M \iff u \in T \text{ in } G.$$

Theorem (RDG 2020)

If $M = \mathrm{Inv}\langle A, t \mid er = 1\rangle$ has decidable word problem then the membership problem for T within $G = \mathrm{Gp}\langle A \mid r = 1\rangle$ is decidable.

Theorem (RDG (2020))

There is a one-relator inverse monoid $\mathrm{Inv}\langle A \mid w = 1\rangle$ with undecidable word problem.

The word problem and groups of units

Key question

For which words $w \in (A \cup A^{-1})^*$ does $\mathrm{Inv}\langle A \mid w = 1\rangle$ have decidable word problem? In particular is the word problem always decidable when w is (a) reduced or (b) cyclically reduced?

Note: A positive answer to (a) would imply the word problem is also decidable for every one-relator monoid $\mathrm{Mon}\langle A \mid u = v\rangle$.

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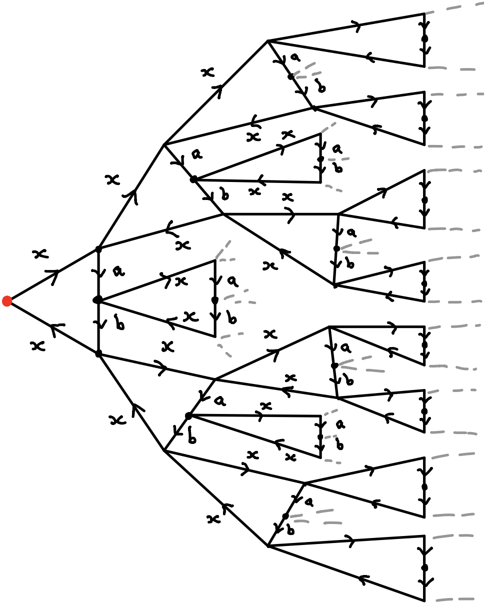
Theorem (Adjan (1966))

The group of units G of a one-relator monoid $M = \mathrm{Mon}\langle A \mid r = 1\rangle$ is a one-relator group. Furthermore, M has decidable word problem.

Problem: What are the groups of units of inverse monoids $\mathrm{Inv}\langle A \mid r = 1\rangle$?

Example - group of units



Theorem (Stephen (1990)) The group of units of $M = \mathrm{Inv}\langle A \mid r = 1\rangle$ is isomorphic to the group $\mathrm{Aut}(S\Gamma(1))$ of label-preserving automorphisms of the Schützenberger graph $S\Gamma(1)$.

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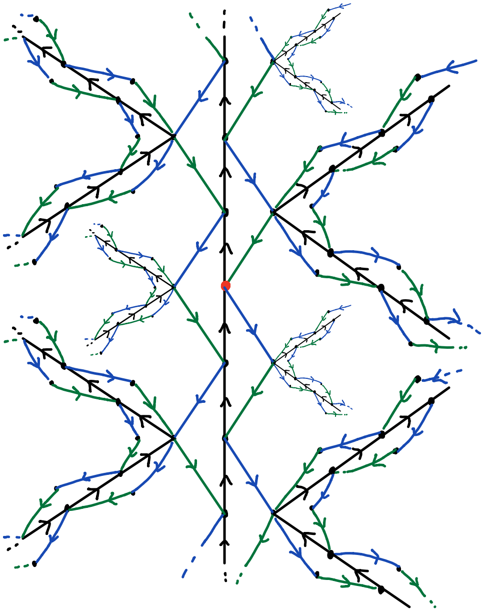

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The group of units is

$$\mathrm{Aut}(S\Gamma(1)) \cong \mathbb{Z}$$

the infinite cyclic group.

Units of one-relator inverse monoids and coherence

Theorem (RDG & Ruškuc (2021))

There exists a one-relator inverse monoid $M = \mathrm{Inv}\langle A \mid r = 1\rangle$ whose group of units G is not a one-relator group.

Question: Is the group of units of $\mathrm{Inv}\langle A \mid r = 1\rangle$ always finitely presented?[2]

[2] It is known to be finitely generated.

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Definition. A finitely presented group G is said to be coherent if every finitely generated subgroup of G is finitely presented.

Open problem (Baumslag (1973))

Is every one-relator group coherent?

Theorem (RDG & Ruškuc (2021))

If all one-relator inverse monoids $\mathrm{Inv}\langle A \mid r = 1 \rangle$ have finitely presented groups of units then all one-relator groups are coherent.

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Suffix membership problem and positive one-relator groups

Definition. The suffix monoid $\mathcal{S}_G$ of $G = \mathrm{Gp}\langle A \mid r = 1\rangle$ is the submonoid generated by the siffixes of r. We say the suffix membership problem is decidable if membership in the submonoid $\mathcal{S}_G$ of G is decidable.

Example
$$G = \mathrm{Gp}\langle x, y \mid x^{-1}yx^2yx^3yx = 1\rangle$$

- Suffix monoid = $\mathrm{Mon}\langle x, yx, xyx, \ldots, yx^2yx^3yx\rangle = \mathrm{Mon}\langle x, yx\rangle$.

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If every $\mathrm{Gp}\langle X \mid x^{-1}yQx = 1\rangle$ with $Q \in X^*$ has decidable suffix membership problem then all monoids $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$ have decidable word problem.

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There is a positive one-relator group $\mathrm{Gp}\langle A \mid w = 1\rangle$, $w \in A^+$, with undecidable submonoid membership problem.

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There is a one-relator group $\mathrm{Gp}\langle A \mid v^{-1}u = 1\rangle$, where $u, v \in A^+$ and $v^{-1}u$ is reduced, with undecidable suffix membership problem.

Open problems

Problem. Let $G = \mathrm{Gp}\langle A \mid r = 1\rangle$. Is membership in $\mathrm{Mon}\langle A\rangle$ decidable? i.e. is there an algorithm that decides if a given word can be written positively?

Problem. Does every group $\mathrm{Gp}\langle X \mid x^{-1}yQx = 1\rangle$ with $Q \in X^*$ have decidable suffix membership problem?

Problem. Classify one-relator groups with decidable submonoid membership problem. It remains open for

- Baumslag–Solitar groups $B(m, n) = \mathrm{Gp}\langle a, b \mid b^{-1}a^m ba^{-n} = 1\rangle$
  - Solved for $BS(1, n)$ by Cadilhac, Chistikov & Zetzsche (2020).
- Surface groups $\mathrm{Gp}\langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid [a_1, b_1] \ldots [a_g, b_g] = 1\rangle$.
- One-relator groups with torsion $\mathrm{Gp}\langle A \mid r^n = 1\rangle$, $n \geq 2$.

Is there a one-relator group that embeds trace monoid of P_4 but not $A(P_4)$?

Problem. Does $\mathrm{Inv}\langle A \mid w = 1\rangle$ have decidable word problem when w is a reduced word?

Problem. Is the group of units of $\mathrm{Inv}\langle A \mid w = 1\rangle$ finitely presented?