Free idempotent generated semigroups over the full linear monoid

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Uppsala, September 2012

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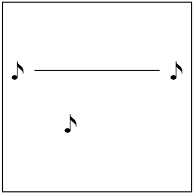
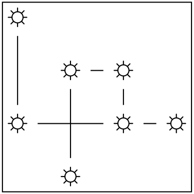
Connectedness in tables

$$M = \begin{pmatrix} \sun & \heartsuit & \smiley & \heartsuit \\ \eighthnote & \sun & \sun & \eighthnote \\ \sun & \eighthnote & \sun & \sun \\ \smiley & \sun & \smiley & \heartsuit \end{pmatrix}$$



$\Delta(\eighthnote)$ is not connected



$\Delta(\sun)$ is connected

Tables in algebra

Multiplication tables

Group multiplication tables

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

- The multiplication table of a group is a Latin square, so..
- None of the graphs $\Delta(x)$ will be connected.

Multiplication table of a field.

Field with three elements $\mathbb{F} = \{0, 1, 2\}$.

	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

- $\Delta(0)$ is connected
- $\Delta(f)$ is not connected for every $f \neq 0$

Tables in algebra

Vectors

$\mathbb{F} = \{0, 1\}$, vectors in $\mathbb{F}^3$, entries in table from $\mathbb{F}$

	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$
(0, 0, 0)	0	0	0	0	0	0	0	0
(0, 0, 1)	0	1	0	1	0	1	0	1
(0, 1, 0)	0	0	1	1	0	0	1	1
(0, 1, 1)	0	1	1	0	0	1	1	0
(1, 0, 0)	0	0	0	0	1	1	1	1
(1, 0, 1)	0	1	0	1	1	0	1	0
(1, 1, 0)	0	0	1	1	1	1	0	0
(1, 1, 1)	0	1	1	0	1	0	0	1

- For every symbol x in the table $\Delta(x)$ is connected.

Outline

- Free idempotent generated semigroups
  - Background
  - Maximal subgroups of free idempotent generated semigroups
- The full linear monoid
  - Basic properties
  - The free idempotent generated semigroup over the full linear monoid
  - Proof ideas: connectedness properties in tables
- Open problems

Free idempotent generated semigroups

S - semigroup, $E = E(S)$ - idempotents $e = e^2$ of S

E carries a certain abstract structure: that of a biordered set

- Nambooripad (1979), Easdown (1985)

General idea: Fix a biorder E and investigate those semigroups whose idempotents carry this fixed biorder structure.

Within this family there is a unique "free" object $IG(E)$ which is the semigroup defined by presentation:

$$IG(E) = \langle E \mid e \cdot f = ef \ \ (e,f \in E, \ \{e,f\} \cap \{ef,fe\} \neq \emptyset) \ \rangle$$

$IG(E)$ is called the free idempotent generated semigroup on E.

First steps towards understanding $IG(E)$

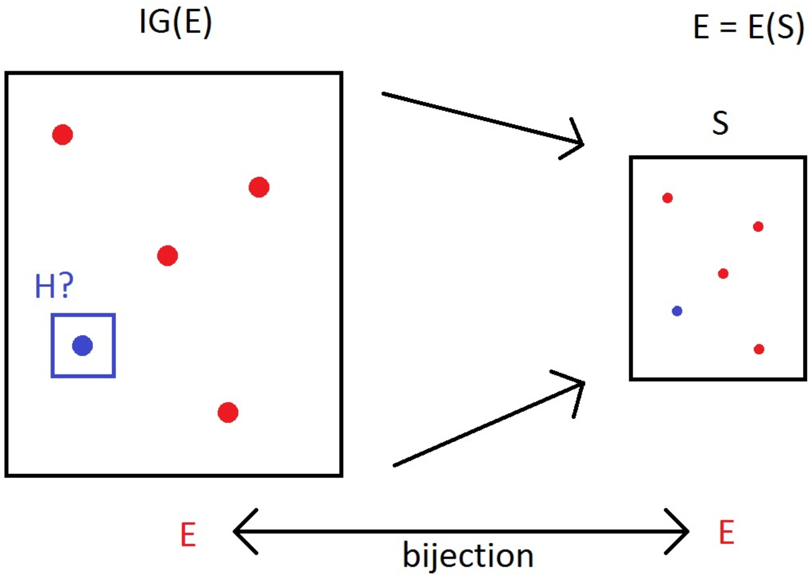
S - semigroup, $E = E(S)$ - idempotents of S

Theorem (Easdown (1985))

Let S be an idempotent generated semigroup with $E = E(S)$. Then $IG(E)$ is an idempotent generated semigroup and there is a surjective homomorphism $\phi : IG(E) \to S$ which is bijective on idempotents.

Conclusion. It is important to understand $IG(E)$ if one is interested in understanding an arbitrary idempotent generated semigroups.

Question. What can be said about the maximal subgroups of free idempotent generated semigroups?



IG(E)
E = E(S)
S
H?
E
bijection
E

The full linear monoid

$\mathbb{F}$ - arbitrary field, $\quad n \in \mathbb{N}$

$$M_n(\mathbb{F}) = \{n \times n \text{ matrices over } \mathbb{F}\}.$$

- Plays an analogous role in semigroup theory as the general linear group does in group theory.
- Important in a range of areas:
  - Representation theory of semigroups
  - Putcha–Renner theory of linear algebraic monoids and finite monoids of Lie type. Here the biordered set of idempotents of the monoid may be viewed as a generalised building, in the sense of Tits.

Aim

Investigate maximal subgroups of $IG(E)$ where $S = M_n(\mathbb{F})$ and $E = E(S)$.

Properties of $M_n(\mathbb{F})$

Theorem (J.A. Erdös (1967))

$$\langle E(M_n(\mathbb{F}))\rangle = \{\textit{identity matrix and all non-invertible matrices}\}.$$

- Greens relations $\mathcal{D}$ and $\mathcal{J}$ coincide, and the $\mathcal{D}$-classes are

$$D_r = \{A : \operatorname{rank}(A) = r\}, \quad r \leq n.$$

- The maximal subgroups in D_r are isomorphic to $GL_r(\mathbb{F})$.

The problem

$\mathbb{F}$ - arbitrary field, $n \in \mathbb{N}$

$$M_n(\mathbb{F}) = \{n \times n \text{ matrices over } \mathbb{F}\}.$$

By Easdown (1985) we may identify

$$E = E(M_n(\mathbb{F})) = E(IG(E)).$$

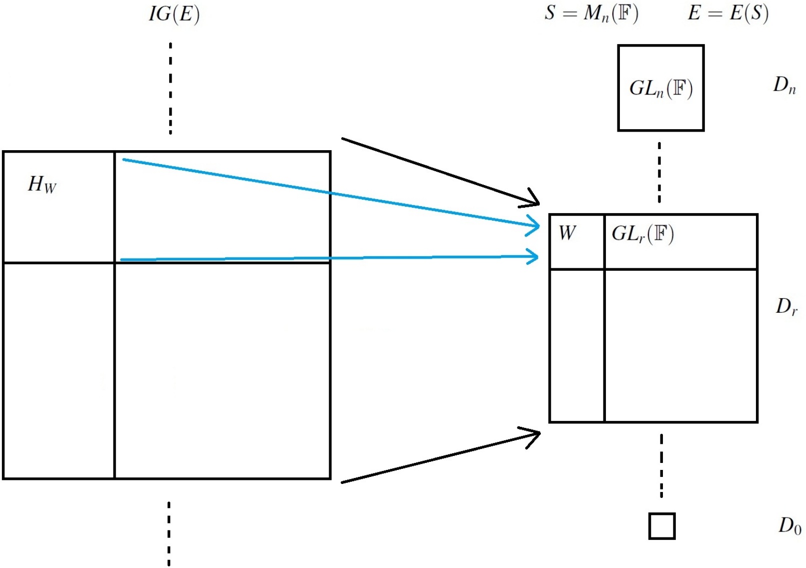
Fix W an idempotent matrix of rank r.

Problem: Identify the maximal subgroup H_W of

$$IG(E) = \langle E \mid e \cdot f = ef \ \ (e,f \in E,\ \{e,f\} \cap \{ef,fe\} \neq \emptyset) \ \rangle$$

containing W.

General fact: H_W is a homomorphic preimage of $GL_r(\mathbb{F})$.



$IG(E)$
$S = M_n(\mathbb{F})$
$E = E(S)$
$GL_n(\mathbb{F})$
D_n
H_W
W
$GL_r(\mathbb{F})$
D_r
D_0

Results

$n \in \mathbb{N}$, $\mathbb{F}$ - field, $E = E(M_n(\mathbb{F}))$,

$W \in M_n(\mathbb{F})$ - idempotent of rank r

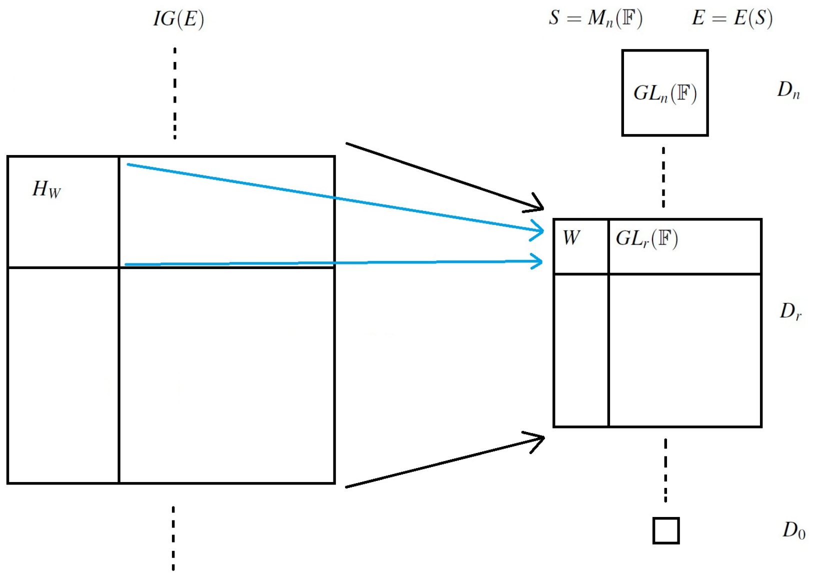
H_W = maximal subgroup of $IG(E)$

Theorem (Brittenham, Margolis, Meakin (2009))

For $n \geq 3$ and $r = 1$ we have $H_W \cong GL_r(\mathbb{F}) \cong \mathbb{F}^*$.

Theorem (Dolinka, Gray (2012))

Let n and r be positive integers with $r < n/3$. Then $H_W \cong GL_r(\mathbb{F})$.

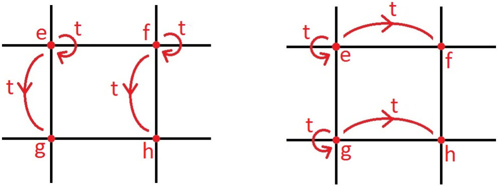


$IG(E)$
$S = M_n(\mathbb{F})$
$E = E(S)$
$GL_n(\mathbb{F})$
D_n
H_W
W
$GL_r(\mathbb{F})$
D_r
D_0

Writing down a presentation for H_W

Singular squares

A singular square is a 2×2 rectangular band inside D_r such that there exists an idempotent $t \in E$ acting on the square in one of the following two ways:



- In many natural examples, including $M_n(\mathbb{F})$, we have

$$\{ \text{ singular squares } \} = \{ \text{ } 2 \times 2 \text{ rectangular bands } \}$$

  (Brittenham, Margolis, Meakin (2010)).
- The group H_W is defined by relations arising from singular squares.
- Conclusion: The group H_W is determined by the structure of the idempotents in D_r.

Step 1: Writing down a presentation for H_W

A Rees matrix representation

Definition

A matrix is in reduced row echelon form (RRE form) if:

- rows with at least one nonzero element are above any rows of all zeros
- the leading coefficient (the first nonzero number from the left) of a nonzero row is always strictly to the right of the leading coefficient of the row above it, and
- every leading coefficient is 1 and is the only nonzero entry in its column.

Examples

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Step 1: Writing down a presentation for H_W

$n, r \in \mathbb{N}$ fixed with $r < n$

$$\begin{aligned} \mathcal{Y}_r &= \{r \times n \text{ rank } r \text{ matrices in RRE form}\} \\ \mathcal{X}_r &= \{\text{transposes of elements of } \mathcal{Y}_r\} \end{aligned}$$

- Matrices in $\mathcal{Y}_r$ have no rows of zeros, so have r leading columns.

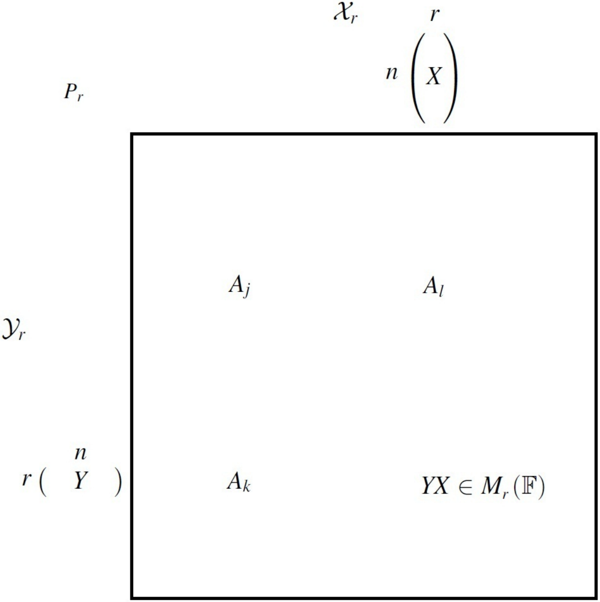
$$\text{e.g. } n = 4, \; r = 3, \; \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{Y}_3.$$

- Define a matrix $P_r = (P_r(Y, X))$ defined for $Y \in \mathcal{Y}_r$, $X \in \mathcal{X}_r$ by

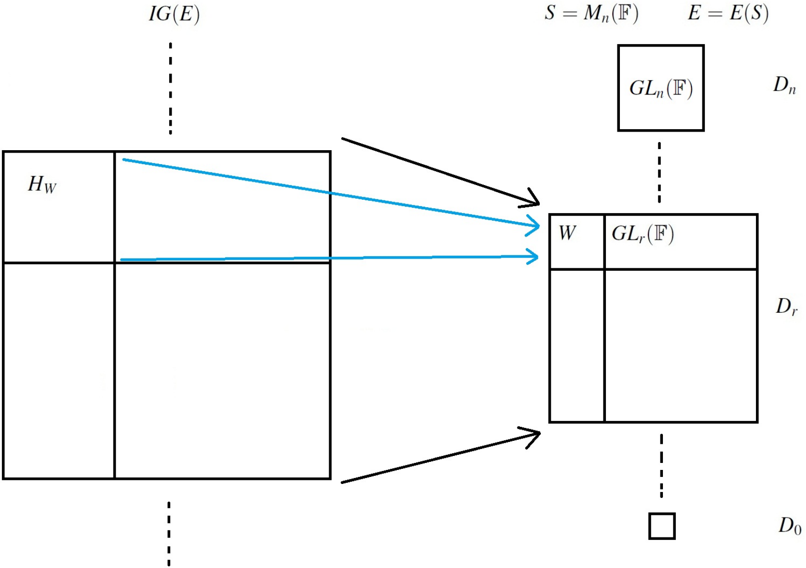
$$P_r(Y, X) = YX \in M_r(\mathbb{F}).$$

Theorem

The principal factor D_r^0 of $M_n(\mathbb{F})$ is isomorphic to the Rees matrix semigroup $\mathcal{M}^0(GL_r(\mathbb{F}); \mathcal{X}_r, \mathcal{Y}_r; C_r)$ where the structure matrix $C_r = (C_r(Y, X))$ is defined by $C_r(Y, X) = YX$ if $YX \in GL_r(\mathbb{F})$ and 0 otherwise.



$\mathcal{X}_r$
r
n
X
P_r
A_j
A_l
$\mathcal{Y}_r$
n
r
Y
A_k
$YX \in M_r(\mathbb{F})$



$IG(E)$
$S = M_n(\mathbb{F})$
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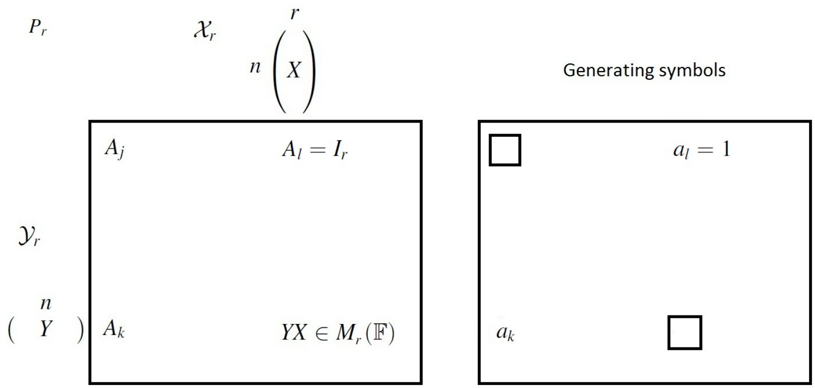
D_n
H_W
W
$GL_r(\mathbb{F})$
D_r
D_0

The group H_W is defined by the presentation with...



Generators: $\{a_j \mid A_j$ is an entry in P_r satisfying $A_j \in GL_r(\mathbb{F})\}$

Relations:

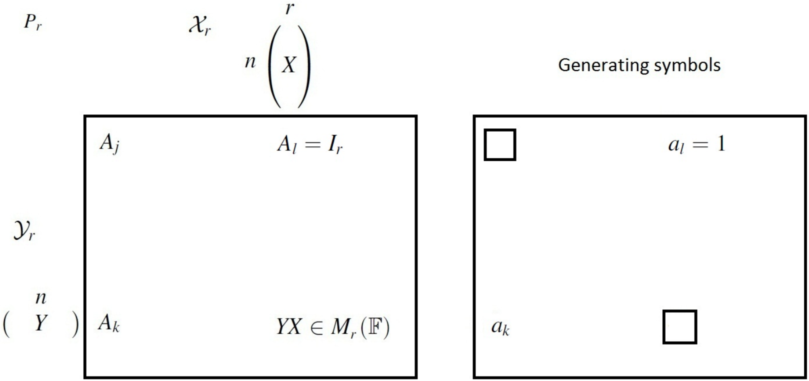
(I) $a_j = 1$ for all entries A_j in P_r satisfying $A_j = I_r$

(II) $a_j a_k^{-1} = a_l a_m^{-1} \Leftrightarrow (A_j, A_k, A_l, A_m)$ is a singular square of invertible $r \times r$ matrices from P_r with $A_j^{-1} A_k = A_l^{-1} A_m$.

A_j A_k

A_l A_m

Structure of the proof that $H_W \cong GL_r(\mathbb{F})$



Step 1: Write down a presentation for H_W.

Step 2: Prove that for any two entries A_j, A_k in the table P_r, if $A_j = A_k \in GL_r(\mathbb{F})$ then $a_j = a_k$ is deducible from the relations.

Step 3: Find defining relations for $GL_r(\mathbb{F})$ using the singular square relations (II).

Step 2: Strong edges and relations

Definition

We say entries A_j and A_k with $A_j = A_k$ are connected by a strong edge if

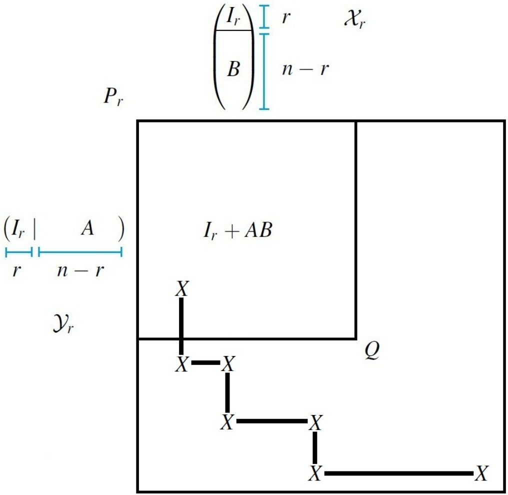
$$\begin{array}{ccc} A_j & \text{———} & A_k \\ & & \\ I_r & & I_r \end{array} \qquad \text{or} \qquad \begin{array}{cc} A_j & I_r \\ \Big| & \\ A_k & I_r \end{array}$$

Lemma: If $A_j = A_k \in GL_r(\mathbb{F})$ are connected by a strong edge then $a_j = a_k$ is a consequence of the relations.

$$\begin{array}{ccc} A_j & \text{———} & A_k \\ & & \\ I_r & & I_r \end{array} \quad \Rightarrow \quad \begin{array}{cc} a_j & a_k \\ & \\ 1 & 1 \end{array} \quad \Rightarrow \; a_j = a_k \text{ can be deduced}$$

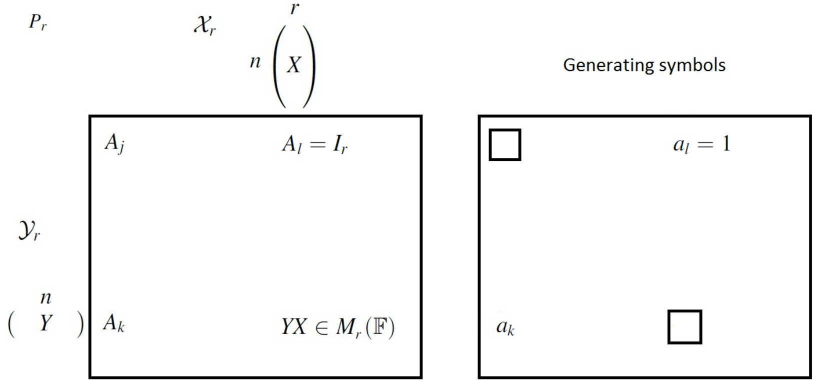
A singular square   Using relations (I)

Proof of Step 2



Proposition: For every pair A_j, A_k of entries in P_r, if $A_j = A_k$ then there is a strong path between A_j and A_k. Thus, for every pair $A_j = A_k \in GL_r(\mathbb{F})$ in the table P_r the relation $a_j = a_k$ is deducible.

Structure of the proof that $H_W \cong GL_r(\mathbb{F})$



Step 1: Write down a presentation for H_W.

Step 2: Prove that for any two entries A_j, A_k in the table P_r, if $A_j = A_k \in GL_r(\mathbb{F})$ then $a_j = a_k$ is deducible from the relations.

Step 3: Find defining relations for $GL_r(\mathbb{F})$ among the singular square relations (II).

Open problems

Theorem (Dolinka, Gray (2012))

Let n and r be positive integers with $r < n/3$. Then $H_W \cong GL_r(\mathbb{F})$.

- What happens in higher ranks?

Conjecture (Brittenham, Margolis, Meakin (2010))

Let n and r be positive integers with $r \leq n/2$. Then $H_W \cong GL_r(\mathbb{F})$.

- The same result might even be true for $r < n - 1$.
- To prove the conjecture we need a better understanding of the combinatorial connectedness properties of the Rees structure matrices of the principal factors of $M_n(\mathbb{F})$.
- We know that the above result about strong connectedness of symbols does not hold for higher ranks.